Unitary groups and augmented Cuntz semigroups of separable simple $\mathcal{Z}$-stable C^*-algebras

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Abstract

Let A be a separable simple exact $\mathcal{Z}$-stable C^*-algebra. We show that the unitary group of $\tilde{A}$ has the cancellation property. If A has continuous scale then the Cuntz semigroup of $\tilde{A}$ has strict comparison property and a weak cancellation property. Let C be a 1-dimensional non-commutative CW complex with $K_1(C)=\{0\}$. Suppose that $\lambda: \mathrm{Cu}^\sim(C)\to \mathrm{Cu}^\sim(A)$ is a morphism in the augmented Cuntz semigroups which is strictly positive. Then there exists a sequence of homomorphisms $\varphi_n: C\to A$ such that $\lim_{n\to\infty}\mathrm{Cu}^\sim(\varphi_n)=\lambda$. This result leads to the proof that every separable amenable simple C^*-algebra in the UCT class has rationally generalized tracial rank at most one.

1 Introduction

Recently there has been some rapid progress in the Elliott program of classification of separable amenable C^*-algebras. For example, all unital separable amenable simple Jiang-Su stable C^*-algebras in the UCT class have been classified up to isomorphisms by the Elliott invariant (see [22], [23], [15], and [38], for example). Let A be a unital $\mathcal{Z}$-stable C^*-algebra, where $\mathcal{Z}$ is the Jiang-Su algebra. It was shown by M. Rørdam ([34]) that A either has stable rank one, i.e., the invertible elements in A are dense in A, or A is purely infinite. As a consequence, in the finite case, by [28] and [29], A has the cancellation of projections and $U(A)/U_0(A)=K_1(A)$. There are other regular properties for $\mathcal{Z}$-stable C^*-algebras (see also [41]). It is these regular properties that make the class of unital separable amenable simple $\mathcal{Z}$-stable C^*-algebras classifiable.

One may expect that non-unital simple $\mathcal{Z}$-stable C^*-algebras have similar properties. Indeed, by M. Rørdam ([34]), non-unital simple $\mathcal{Z}$-stable C^*-algebras also have strict comparison for positive elements and nice picture of Cuntz semigroups (see [18]). It is shown by L. Robert ([31]) that, if A is a simple stably projectionless $\mathcal{Z}$-stable C^*-algebra, then A has almost stable rank one, i.e., the invertible elements in $\tilde{A}$, the unitization of A, are dense in A. If A is a separable simple $\mathcal{Z}$-stable C^*-algebra which is not stably projectionless, then A must have stable rank one. So we will mainly consider stably projectionless simple C^*-algebras. There is a fundamental difference between unital simple C^*-algebras and stably projectionless simple C^*-algebras. In [20], we show that there is a unique separable amenable simple stably projectionless $\mathcal{Z}$-stable C^*-algebra $\mathcal{Z}_0$ in the UCT class with a unique tracial state such that $K_i(\mathcal{Z}_0)=K_i(\mathbb{C})$ $(i=0,1)$. Let A be any finite separable amenable simple C^*-algebra. Then $A\otimes\mathcal{Z}_0$ is a separable amenable simple stably projectionless $\mathcal{Z}$-stable C^*-algebra such that $K_i(A\otimes\mathcal{Z}_0)=K_i(A)$ $(i=0,1)$ and $T(A\otimes\mathcal{Z}_0)=T(A)$. This means that there is a rich class of separable simple amenable stably projectionless $\mathcal{Z}$-stable C^*-algebras. There are also separable amenable stably projectionless simple C^*-algebras which cannot be written as $A\otimes\mathcal{Z}_0$ for any separable simple amenable C^*-algebra A (see [21]).

More recently a classification theorem for non-unital separable simple amenable $\mathcal{Z}$-stable C^*-algebras with stable rank one in the UCT class was presented in the original version of [21]. The motivation of this note is to provide a technical result that removes the condition of stable rank one. We will not, however, prove that, in general, a separable simple stably projectionless $\mathcal{Z}$-stable C^*-algebra has stable rank one. Instead, we will show that these C^*-algebras have nice properties which will lead to a reduction theorem, i.e., every separable amenable simple stably projectionless C^*-algebra in the UCT class has rationally generalized tracial rank one without assuming that $A \otimes Q$ has stable rank one. Therefore, as in [21], the additional condition of stable rank one in the classification theorem mentioned above is removed.

We begin with the question whether a non-unital separable simple $\mathcal{Z}$-stable C^*-algebra A still has the cancellation of projections for $\tilde{A}$ and the property $U(\tilde{A})/U_0(\tilde{A}) = K_1(A)$. In this note, we first show that, indeed, $U(\tilde{A})/U_0(\tilde{A}) = K_1(A)$ (see Corollary 3.8).

One notices that we study the unitary group of $M_n(\tilde{A})$ not that of $M_n(A)$ as $M_n(A)$ has no unitaries. Naturally we study the Cuntz semigroup $\mathrm{Cu}(\tilde{A})$ of $\tilde{A}$, not $\mathrm{Cu}(A)$ when A is stably projectionless but $K_0(A) \neq \{0\}$. To make the strict comparison more meaningful, we assume that A has continuous scale. It should be noted that $\tilde{A}$ is not $\mathcal{Z}$-stable and we do not know whether $\tilde{A}$ has stable rank one. We do not even know whether $\mathrm{Cu}(\tilde{A})$ has cancellation of projections. Nevertheless, we will show that, in the case that A has continuous scale, indeed, $\mathrm{Cu}(\tilde{A})$ has the strict comparison and a weak cancellation property. These two aforementioned properties (one for K_1 and one for $\mathrm{Cu}(\tilde{A})$) are proved without assuming A has stable rank one.

L. Robert shows ([30]) that the augmented Cuntz semigroup $\mathrm{Cu}^\sim$ classifies homomorphisms from 1-dimensional noncommutative CW complexes with trivial K_1-groups to C^*-algebras of stable rank one. This result played an important role in the proof of the fact that every unital separable finite simple C^*-algebra with finite nuclear dimension in the UCT class has rationally generalized tracial rank at most one. Since unital separable simple amenable $\mathcal{Z}$-stable C^*-algebras in the UCT class with rationally generalized tracial rank at most one are previously shown to be classified by the Elliott invariant, this latter result leads to the classification of all unital separable simple amenable C^*-algebras of finite nuclear dimension in the UCT class (see [15]).

The additional condition that C^*-algebras have stable rank one in the classification results for non-unital simple C^*-algebras mentioned above was used to apply the following existence result of L. Robert ([30]): Let C be a 1-dimensional noncommutative CW complex with $K_1(C) = \{0\}$ and with a strictly positive element e_C. If $\lambda : \mathrm{Cu}^\sim(C) \to \mathrm{Cu}^\sim(A)$ is a morphism in **Cu** with $\lambda([e_C]) \leq [a]$ for some $a \in A_+$, then there exists a homomorphism $\varphi : C \to A$ such that $\mathrm{Cu}^\sim(\varphi) = \lambda$.

As mentioned in the abstract, we show, without assuming A has stable rank one, that there is a sequence of homomorphisms $\varphi_k : C \to A$ such that $\lim_{k\to\infty} \mathrm{Cu}^\sim(\varphi_k) = \lambda$, if, in addition, A is an exact separable simple $\mathcal{Z}$-stable C^*-algebra and $\lambda([c]) \neq 0$ for any $c \in C_+ \setminus \{0\}$ (see Definition 5.1). It turns out that this weaker version of existence theorem will be sufficient for the purpose of proving that every separable simple amenable stably projectionless C^*-algebra in the UCT class has rationally generalized tracial rank at most one. Therefore, we are able to remove the redundant condition of stable rank one in the original version of [21]. Together with the classification theorem in [21], every finite separable simple amenable $\mathcal{Z}$-stable C^*-algebra in the UCT class, in fact, has stable rank one.

Let C be a 1-dimensional NCCW complex. L. Robert shows that there are 1-dimensional NCCW complexes $C_0, C_1, ..., C_n$ such that $C_0 = C_0((0,1])$, $C_n = C$, C_i is either stably isomorphic to C_{i-1}, or C_i is the unitization of C_{i-1}, or C_{i-1} is the unitization of C_i, $i = 1, 2, ..., n$. Let B be a separable simple stably projectionless $\mathcal{Z}$-stable C^*-algebra. Then B has almost stable rank one. We first show that, for $C = C_0$, a homomorphism h can be produced so that $\mathrm{Cu}^\sim(h)$ will be the given λ. We then show our approximate version of existence theorem holds for C^*-algebras C_1 and beyond. However, this process requires to change the target algebra B to $M_n(\tilde{B})$ (for any integer $n \geq 1$). The problem is that we do not know whether $M_n(\tilde{B})$ has stable rank one.

Let $\varphi, \psi : C \to M_n(\tilde{B})$ be homomorphisms such that $\mathrm{Cu}^\sim(\varphi) = \mathrm{Cu}^\sim(\psi)$. Suppose that $e \in M_k(C)$ is a nonzero projection and $p = \varphi(e)$ and $q = \psi(e)$. Note that $[p] = [q]$ in $\in \mathrm{Cu}^\sim(\tilde{B})$ if and only if there is an integer $1 \le m\ (\le 2)$ such that $p \oplus 1_m \sim q \oplus 1_m$ in the Cuntz semigroup of $\tilde{B}$. However, the classification of homomorphisms by $\mathrm{Cu}^\sim$ is not possible without $p \sim q$. We will not attempt to prove that the functor $\mathrm{Cu}^\sim$ (introduced by Robert) classifies homomorphisms from 1-dimensional noncommutative CW complexes. The existence part of Theorem 1.0.1 of [30] depends on the uniqueness part of that. Nevertheless, we will find a way to circumvent this to obtain an approximate version of existence theorem without the uniqueness theorem.

The paper is organized as follows: In section 2, we list some basics regarding the notion of almost stable rank one and other notations. In section 3, we show that, with slightly more general assumption, $U(\tilde{A})/U_0(\tilde{A}) = K_1(A)$ for any separable simple $\mathcal{Z}$-stable C^*-algebra A. In section 4, we present some crucial technical statements about comparison in $M_n(\tilde{A})$ (for any integer $n \ge 1$) involving unitaries. We show that $\mathrm{Cu}(\tilde{A})$ has the strict comparison and a weak cancellation when A has continuous scale. In section 5, we start some discussion of approximation in augmented Cuntz semigroups and perturbation of homomorphisms. In section 6, we deal with unitization. Finally we present the main results in section 7.

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2 Basics

Definition 2.1 ([31])**.** If C is a unital C^*-algebra, let $GL(C)$ be the group of invertible elements of C. A C^*-algebra A is said to have almost stable rank one, if $GL(\tilde{B})$ is dense in B for every hereditary C^*-subalgebra B of A, where $\tilde{B}$ is the unitization of B, if B is not unital.

If A has almost stable rank one, by the definition, every hereditary C^*-subalgebra of A has almost stable rank one.

For a separable simple C^*-algebra A, if A does not have stable rank one, but has almost stable rank one, then A must be projectionless, by the following observation which is known.

Proposition 2.2. *Let A be a σ-unital C^*-algebra which has almost stable rank one. Then A has stable rank one, if A has a nonzero full projection. If A is simple and $M_n(A)$ has almost stable rank one for each n, then A either has stable rank one, or A is stably projectionless.*

Moreover, if A is simple and has almost stable rank one, then $\mathrm{Ped}(A)$*, the Pedersen ideal of A, has no infinite elements (see Definition 1.1 of [24]), and $\tilde{A}$ is finite.*

Proof. Fix an integer $n \ge 1$. Suppose that $M_n(A)$ has almost stable rank one. Let $p \in M_n(A)$ be a nonzero full projection. By the definition, the invertible elements of $pM_n(A)p$ is dense in $pM_n(A)p$. So $pM_n(A)p$ has stable rank one. By [6], since A is σ-unital, $pM_n(A)p$ is stably isomorphic to A. Therefore A has stable rank one. Note that the above works for $n = 1$. This proves the first part of the statement.

Suppose that A is simple and has almost stable rank one. If A has stable rank one, $\tilde{A}$ has stable rank one. Then A and $\tilde{A}$ are stably finite. In particular, $\mathrm{Ped}(A)$ has no infinite elements. Now suppose that A does not have stable rank one but has almost stable rank one. If $\mathrm{Ped}(A)$ has an infinite element, by Theorem 1.2 of [24], A has a non-trivial projection. By the first part of the proposition, A has stable rank one. A contradiction.

If $\tilde{A}$ is not finite, there is $v \in \tilde{A}$ such that $vv^* = 1$ and $v^*v \ne 1$. Then $1 - v^*v \in A$ is a non-zero projection. By what has been proved, this would imply that A has stable rank one. □

The following is a non-unital version of a result of Rørdam ([34]) which follows from Theorem 6.7 of [34] and a result of L. Robert (Theorem 1.2 [31]).

Theorem 2.3. *Let A be a σ-unital simple $\mathcal{Z}$-stable C^*-algebra. Then one and only one of the following must occur:*

(1) A is purely infinite,

(2) A has stable rank one,

(3) A does not have stable rank one, but has almost stable rank one and is stably projectionless. Moreover A has a non-zero 2-quasitrace.

Proof. Suppose that neither does A have stable rank one, nor A is purely infinite.

Note, since A is $\mathcal{Z}$-stable, so is $M_n(A)$ for each $n \in \mathbb{N}$. If $M_n(A)$ contains a non-zero projection p for some $n \in \mathbb{N}$, then $B := pM_n(A)p$, as a unital hereditary C^*-subalgebra, is also $\mathcal{Z}$-stable (Corollary 3.1 of [39]). By Theorem 4.5 of [34], $W(B)$ is almost unperforated. If B does not have stable rank one, then, by Corollary 3.6 of [7], $M_k(B)$ does not have stable rank one for any k. Therefore, by Theorem 6.7 of [34], none of $M_k(B)$ are finite. Then, by Corollary 5.1 of [34] (see also Proposition 4.9 of [19]), B is purely infinite. By the assumption at the very beginning, $M_n(A)$ has no non-zero projection for all n. In other words, A is stably projectionless. Then, by [31], A has almost stable rank one. Moreover, by Corollary 5.1 of [34] (see Proposition 4.9 of [19]), A has a non-zero 2-quasitrace. $\square$

We do not know, at the moment, that case (3) of Theorem 2.3 can actually occur.

Proposition 2.4. *Let A be a C^*-algebra which has almost stable rank one. Then, for any integer $n \geq 1$, $GL(M_n(\tilde{A}))$ is dense in $M_n(A)$. Moreover, $GL((\tilde{A} \otimes \mathcal{K})^{\sim})$ is dense in $A \otimes \mathcal{K}$.*

Proof. We prove the first part by induction. Suppose that $GL(M_n(\tilde{A}))$ is dense in $M_n(A)$. We will show that $GL(M_{n+1}(\tilde{A}))$ is dense in $M_{n+1}(A)$.

Let $x \in M_{n+1}(A)$. Put $p := \mathrm{diag}(1_{\tilde{A}}, \overbrace{0, 0, ..., 0}^{n})$. Let $a = pxp$, $b = (1-p)x(1-p)$, $c = px(1-p)$ and $d = (1-p)xp$. Hence we may write

$$x = \begin{pmatrix} a & c \\ d & b \end{pmatrix}. \tag{e 2.1}$$

Let $\varepsilon > 0$. By the inductive assumption, there is $b' \in GL(M_n(\tilde{A}))$ such that $\|b - b'\| < \varepsilon$. Note

$$c(b')^{-1}d = px(1-p)(b')^{-1}(1-p)xp \in p(M_n(A))p\,(= A). \tag{e 2.2}$$

Therefore (since A has almost stable rank one) there is $z \in GL(\tilde{A})$ such that $\|z-(a-c(b')^{-1}d)\| < \varepsilon$. Set $a' = z + c(b')^{-1}d \in \tilde{A}$. Then

$$\|a - a'\| = \|a - c(b')^{-1}d - z\| < \varepsilon. \tag{e 2.3}$$

Moreover, $a' - c(b')^{-1}d = z$. Put

$$y = a' + b' + c + d = \begin{pmatrix} a' & c \\ d & b' \end{pmatrix}.$$

Then $y \in M_{n+1}(\tilde{A})$ and $\|x-y\| < \varepsilon$. It follows from Lemma 3.1.5 of [25] (see the proof of Lemma 3.4 of [28]) that y is invertible.

For the last part, let $a \in A \otimes \mathcal{K}$ and $1 > \varepsilon > 0$. Viewing $M_n(A)$ as a C^*-subalgebra of $A \otimes \mathcal{K}$, we may assume that $a \in M_n(A)$ for some large n. By what has been proved, we have an invertible element $b \in M_n(\tilde{A})$ such that $\|b - a\| < \varepsilon$. Write $b = (c_{i,j})_{n\times n}$ with $c_{i,j} = \alpha_{i,j} + a_{i,j}$, where $\alpha_{i,j} \in \mathbb{C}$ and $a_{i,j} \in A$. Let E_n be the identity of $M_n(\tilde{A})$. Put $x := b + \varepsilon \cdot (1_{(A\otimes\mathcal{K})^{\sim}} - E_n)$. Then $x \in GL((\tilde{A} \otimes \mathcal{K})^{\sim})$ and $\|a - x\| < \varepsilon$. $\square$

Definition 2.5. Let A be a C^*-algebra. Denote by $A^{\mathbf{1}}$ the unit ball of A. Let $a \in A_+$. Denote by $\mathrm{Her}(a)$ the hereditary C^*-subalgebra $\overline{aAa}$. If $a, b \in A_+$, we write $a \lesssim b$, (a is Cuntz smaller than b), if there exists a sequence of $x_n \in A$ such that $a = \lim_{n\to\infty} x_n^* x_n$ and $x_n x_n^* \in \mathrm{Her}(b)$. If $a \lesssim b$ and $b \lesssim a$, then we say a is Cuntz equivalent to b and write $a \sim b$. The Cuntz equivalence class represented by a will be denoted by $[a]$. So we write $[a] \le [b]$, if $a \lesssim b$. Also $[a] \ll [b]$ means that, if for any increasing sequence $\{x_n\}$ such that $[b] \le \sup_n x_n$, then $[a] \le x_n$ for some n. It is well known that, for any $0 < \varepsilon < \|a\|$, $[(a-\varepsilon)_+] \ll [a]$ (see the middle of the proof of Lemma 2.1.1 of [30] and Theorem 1 of [11]). Denote by $\mathrm{Cu}(A)$ the Cuntz semigroup of A (equivalence classes in $A \otimes \mathcal{K}$). An element $x \in \mathrm{Cu}(A)$ is *compact*, if $x \ll x$. In what follows, we will also use the augmented semigroup $\mathrm{Cu}^\sim(A)$ introduced in [30] and the revised version in [32]. We refer the reader to [30] and [32] for details of the definition of $\mathrm{Cu}^\sim$ and the related terminologies.

Definition 2.6. Let A be a C^*-algebra. Denote by $QT(A)$ the set of 2-quasitraces of A with norm 1, and by $T(A)$ the tracial state space of A. Both could be empty in general.

For any (non-unital) separable C^*-algebra A, denote by $\mathrm{Ped}(A)$ the Pedersen ideal of A. Suppose that B is a full hereditary C^*-subalgebra of A such that $B \subset \mathrm{Ped}(A)$. If $\tau \in QT(B)$, we will continue to write τ for $\tau \otimes \mathrm{Tr}$, where Tr is the densely defined trace on $\mathcal{K}$. We write $QT_0(B)$ for the set of all 2-quasitraces of B with the norm at most one. Since A is stably isomorphic to B, $\tau \in QT_0(B)$ gives a densely defined 2-quasitrace of A. Denote by $\widetilde{QT}(A)$ the set of all densely defined 2-quasitraces on A with the topology given in [18] (see the paragraph above Theorem 4.4 of [18]). In most cases, we will consider only those C^*-algebras with the property that every 2-quasitrace is a trace, for example, A is exact.

If $\tau \in \widetilde{QT}(A)$, we will also continue to write τ on $A \otimes \mathcal{K}$ for $\tau \otimes \mathrm{Tr}$, where Tr is the standard (densely defined) trace on $\mathcal{K}$. So we will view $\widetilde{QT}(A)$ the set of densely defined 2-quasitraces on $A \otimes \mathcal{K}$.

Definition 2.7. Let S be a convex subset of a convex topological cone (which has zero) (such as $\widetilde{QT}(A)$). Let $\mathrm{Aff}(S)$ be the set of all real valued continuous affine functions on S with the property that, if $0 \in S$, then $f(0) = 0$. Define (see [30])

$$\begin{aligned}
\mathrm{Aff}_+(S) &= \{f : \mathrm{Aff}(S) : f(\tau) > 0 \text{ for } \tau \neq 0\} \cup \{0\}, &\text{(e 2.4)}\\
\mathrm{LAff}_+(S) &= \{f : S \to [0,\infty] : \exists \{f_n\}, f_n \nearrow f,\ f_n \in \mathrm{Aff}_+(S)\} \text{ and} &\text{(e 2.5)}\\
\mathrm{LAff}_+^\sim(S) &= \{f_1 - f_2 : f_1 \in \mathrm{LAff}_+(S) \text{ and } f_2 \in \mathrm{Aff}_+(S)\}. &\text{(e 2.6)}
\end{aligned}$$

Note that $0 \in \mathrm{LAff}_+(S)$. For the most part of this paper, $S = T(A)$, or $S = \widetilde{QT}(A)$ in the above definition will be used. In particular, if $S = \widetilde{QT}(A)$ and $f \in \mathrm{LAff}_+(S)$, then $f(0) = 0$.

Definition 2.8. For any $\varepsilon > 0$, define $f_\varepsilon \in C([0,\infty))_+$ by $f_\varepsilon(t) = 0$ if $t \in [0, \varepsilon/2]$, $f_\varepsilon(t) = 1$ if $t \in [\varepsilon, \infty)$ and $f_\varepsilon(t)$ is linear in $(\varepsilon/2, \varepsilon)$.

Let A be a C^*-algebra and τ be in $\widetilde{QT}(A)$. For each $a \in A_+$ define $d_\tau(a) = \lim_{\varepsilon\to 0} \tau(f_\varepsilon(a))$. Note that $f_\varepsilon(a) \in \mathrm{Ped}(A)$ for all $a \in A_+$. Recall that A is said to have the Blackadar strict comparison for positive elements, if $a, b \in (A \otimes \mathcal{K})_+$, then $a \lesssim b$ whenever $d_\tau(a) < d_\tau(b)$ for all non-zero $\tau \in \widetilde{QT}(A)$.

Let A be a separable stably finite simple C^*-algebra. There is an order preserving homomorphism $\iota : \mathrm{Cu}(A) \to \mathrm{LAff}_+(\widetilde{QT}(A))$ defined by $\iota([a]) = d_\tau(a)$ for all $\tau \in \widetilde{QT}(A)$ and for all $a \in (A \otimes \mathcal{K})_+$. Let $\mathrm{Cu}_+(A)$ be the sub-semigroup of those elements in $\mathrm{Cu}(A)$ which cannot be represented by non-zero projections (see Proposition 6.4 of [18]) and let $V(A)$ be the Murray-von Neumann equivalence classes of projections in $A \otimes \mathcal{K}$. If A has the strict comparison, $\iota|_{\mathrm{Cu}_+(A)}$ is surjective and is an order isomorphism, following Corollary 6.8 of [18], we write $\mathrm{Cu}(A) = (V(A)\setminus\{0\}) \sqcup \mathrm{LAff}_+(\widetilde{QT}(A))$, where the mixed addition and the order are defined in the paragraph above Corollary 6.8 of [18] (see also page 10 of [37]). In particular, if $x \in V(A) \setminus \{0\}$ and $y \in \mathrm{LAff}_+(\widetilde{QT}(A))$, then $x + y = x$ if $y = 0$, and $x + y = \iota(x) + y$, if $y \neq 0$, and, $x \leq y$, if $\iota(x)(t) < y(t)$ for all $t \neq 0$, and $y \leq x$, if $y \leq \iota(x)$.

Definition 2.9. A separable simple C^*-algebra A is said to be *regular*, if A is purely infinite, or if A has almost stable rank one and $\mathrm{Cu}(A) = (V(A) \setminus \{0\}) \sqcup \mathrm{LAff}_+(\widetilde{QT}(A))$ (see 2.8 above). By [6], for any non-zero hereditary C^*-subalgebra B of A, $B \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. Therefore $\mathrm{Cu}(B) = \mathrm{Cu}(A)$. Hence, if A is a separable regular simple C^*-algebra, then every non-zero hereditary C^*-subalgebra of A is regular (see the last paragraph of 2.6). Except 3.7 and 3.8, we only consider the case that A is finite. By [31] (also Theorem 2.3 above) and Theorem 6.6 of [18], if A is a separable simple $\mathcal{Z}$-stable C^*-algebra, then A is regular. Recall that a separable simple C^*-algebra is said to be pure (introduced by Winter in [41] with non-unital version in subsection 6.3 of [32]) if $\mathrm{Cu}(A)$ is almost unperforated and almost divisible. While it is not used in this paper, we would like to mention that a finite regular simple C^*-algebra is pure, and, a separable simple C^*-algebra which has almost stable rank one is regular if and only if it is pure as shown in subsection 6.3 of [32] (see also Theorem 6.2 of [37] and Corollary 5.8 of [18], and I.1.4 of [1]). We use the term "regular" only for the convenience here.

We would like to state the following version of a result of Rørdam. Note that we do not assume that $M_n(A)$ has almost stable rank one.

Lemma 2.10. *Let A be a C^*-algebra which has almost stable rank one, a and $b \in M_n(A)_+$ for some integer $n \geq 1$ (or in $(A \otimes \mathcal{K})_+$). Suppose that $a \lesssim b$, then, for any $\varepsilon > 0$, there is a unitary $U \in M_n(\tilde{A})$ (or $U \in \widetilde{(\tilde{A} \otimes \mathcal{K})}$) such that*

$$U^* f_\varepsilon(a) U \in \mathrm{Her}(b). \tag{e 2.7}$$

Proof. By Proposition 2.4, $GL(M_n(\tilde{A}))$ is dense in $M_n(A)$ (or $GL(\widetilde{(\tilde{A} \otimes \mathcal{K})})$ is dense in $A \otimes \mathcal{K}$). Then the proof of (iv) $\Rightarrow$ (v) in Proposition 2.4 of [33] (applying Theorem 5 of [27]) works here. ☐

The following is taken from the proof of 1.5 of [26]. But it is also known (see [31]).

Lemma 2.11. *Let A be a C^*-algebra which has almost stable rank one. Suppose that $a \in (A \otimes \mathcal{K})_+$ (or $a \in A_+$), $b \in A_+$, and $a \lesssim b$ in $\mathrm{Cu}(A)$. Suppose that $1/4 > \varepsilon > 0$ and $f_{\varepsilon/4}(a) \in \mathrm{Her}(b)$. Then, for any $0 < \eta < \varepsilon/4$, there is a unitary $u \in \widetilde{(\tilde{A} \otimes \mathcal{K})}$ (or $u \in \tilde{A}$) such that $u f_\eta(a) u^* \in \mathrm{Her}(b)$ and $u f_\varepsilon(a) = f_\varepsilon(a)$. Moreover, there is a partial isometry $v \in (A \otimes \mathcal{K})^{**}$ (or $v \in A^{**}$) such that $vc, cv^* \in A \otimes \mathcal{K}$ (or in A) for all $c \in \mathrm{Her}(a)$, $vav^* \in \mathrm{Her}(b)$ and $v f_\varepsilon(a) = f_\varepsilon(a)$.*

Furthermore, without assuming $f_{\varepsilon/4}(a) \in \mathrm{Her}(b)$, there is also a partial isometry $v \in (A \otimes \mathcal{K})^{**}$ (or $v \in A^{**}$) such that $vc, cv^* \in A \otimes \mathcal{K}$ (or in A), $v^*vc = c = cv^*v$ and $vcv^* \in \mathrm{Her}(b)$ for all $c \in \mathrm{Her}(a)$.

Proof. There is a unitary $w_1 \in \widetilde{(\tilde{A} \otimes \mathcal{K})}$ (or $w_1 \in \tilde{A}$) such that $b_1 := w_1 f_{\eta/8}(a) w_1^* \in \mathrm{Her}(b)$. By Lemma 2.10. Denote $a_1 := w_1 f_{\varepsilon/4}(a) w_1^* \in \mathrm{Her}(b)$. Note that $a_1 b_1 = a_1$. Therefore

$$\|(b_1 - 1) w_1 f_{\varepsilon/4}(a)\| = \|(b_1 - 1) w_1 f_{\varepsilon/4}(a) w_1^*\| = 0. \tag{e 2.8}$$

In other words, $b_1 w_1 f_{\varepsilon/4}(a)^{1/2} = w_1 f_{\varepsilon/4}(a)^{1/2}$. It follows that $y_1 := w_1 f_{\varepsilon/4}(a)^{1/2} \in \mathrm{Her}(b)$. Moreover,

$$y_1^* y_1 = f_{\varepsilon/4}(a) \text{ and } y_1 y_1^* = w_1 f_{\varepsilon/4}(a) w_1^*. \tag{e 2.9}$$

In what follows, for any $d \in A_+^{\mathbf{1}}$ and $1 > \delta > 0$, $e_\delta(d)$ denotes the open spectral projection of d associated with the interval $(\delta, 1]$. Since $\mathrm{Her}(b)$ has almost stable rank one, by Theorem 5 of [27], there is a unitary $z_1 \in \widetilde{\mathrm{Her}(b)}$ such that

$$z_1 e_{1/4}(|y_1|) = w_1 e_{1/4}(|y_1|) = w_1(f_{\varepsilon/4}(a)^{1/2}). \tag{e 2.10}$$

Note that

$$e_{1/4}(|y_1|) = e_{1/4}(f_{\varepsilon/4}(a)^{1/2}) = e_{\delta_1}(a) \tag{e 2.11}$$

for some $\delta_1 \in (\varepsilon/8, \varepsilon/4)$. By (e 2.10) and (e 2.11),

$$z_1^* w_1 e_{\delta_1}(a) = z_1^*(z_1 e_{1/4}(|y_1|)) = e_{1/4}(|y_1|) = e_{\delta_1}(a). \tag{e 2.12}$$

Write $z_1 = \alpha \cdot 1_{\widetilde{\mathrm{Her}(b)}} + b'$ for some $b' \in \mathrm{Her}(b)$. Replacing z_1 by $\alpha \cdot 1 + b'$, we may view z_1 as a unitary in $(\widetilde{\tilde{A} \otimes \mathcal{K}})$ (or in $\tilde{A}$). Put $u_1 := z_1^* w_1 \in (\widetilde{\tilde{A} \otimes \mathcal{K}})$ (or $u_1 \in \tilde{A}$). Then, for any $x \in \overline{f_{2\delta_1}(a)(A \otimes \mathcal{K})}$, by (e 2.12), $u_1 x = u_1 e_{\delta_1}(|y_1|)x = z_1^* w_1 e_{\delta_1}(a)x = e_{\delta_1}(a)x = x$. In particular, $u_1 f_\varepsilon(a) = f_\varepsilon(a)$. We also have, since $z_1 \in \mathrm{Her}(b)^\sim$,

$$u_1 f_\eta(a) u_1^* = z_1^*(w_1 f_\eta(a) w_1^*) z_1 \le z_1^* b z_1 \in \mathrm{Her}(b). \tag{e 2.13}$$

This proves the first part of the lemma.

To see the second part of the lemma, let $\eta_n = \varepsilon/4^{n+1}$. By virtue of the first part of the lemma, we obtain a sequence of unitaries $\{u_n\} \subset (\widetilde{\tilde{A} \otimes \mathcal{K}})$ (or in $\tilde{A}$) such that

$$u_n b_{n-1} u_n^* \in \mathrm{Her}(b), u_n x = x \text{ for all } x \in \mathrm{Her}(b_{n-1}), \tag{e 2.14}$$

where $b_0 = f_\varepsilon(a)$, $b_n = u_n f_{\eta_n}(b_{n-1}) u_n^*$ for $n = 1, 2, ...$ Note

$$\|u_{n+1}(u_n \cdots u_1 f_{\eta_n}(a) - (u_n \cdots u_1 f_{\eta_n}(a)))\| = \|(u_{n+1} - 1)(u_n \cdots u_1 f_{\eta_n}(b)\| \tag{e 2.15}$$

$$= \|(u_{n+1} - 1)(u_n \cdots u_1) f_{\eta_n}(b)(u_1^* \cdots u_n^*)\| = \|(u_{n+1} - 1) b_n\| = 0. \tag{e 2.16}$$

In other words, $u_{n+1} u_n \cdots u_1 f_{\eta_n}(a) = u_n \cdots u_1 f_{\eta_n}(a)$. Moreover, $u_{n+1} u_n \cdots u_1 f_\varepsilon(a) = f_\varepsilon(a)$ for all n. It follows that $\lim_{n\to\infty} u_{n+1} u_n \cdots u_1 x$ converges in norm for all $x \in \mathrm{Her}(a)$ and $\lim_{n\to\infty} u_{n+1} u_n \cdots u_1 x u_1^* \cdots u_n^* u_{n+1}^*$ converges in norm to an element in $\mathrm{Her}(b)$. Choose a strictly positive x of $\mathrm{Her}(a)_+$ with $\|x\| = 1$ and $x f_\varepsilon(a) = f_\varepsilon(a)$. Let $z = \lim_{n\to\infty} u_{n+1} u_n \cdots u_1 x \in A$. Then $zz^* = \lim_{n\to\infty} u_{n+1} u_n \cdots u_1 x^2 u_1^* \cdots u_n^* u_{n+1}^* \in \mathrm{Her}(b)$. Let $z = v x^{1/2}$ be the polar decomposition in $(A \otimes \mathcal{K})^{**}$ (or in A^{**}). Then v is a partial isometry and, since x is a strictly positive element of $\mathrm{Her}(a)$, $vc, cv^* \in A$, $v^*vc = c = cv^*v$, $vcv^* \in \mathrm{Her}(b)$ for all $c \in \mathrm{Her}(a)$, and

$$v f_\varepsilon(a) = v x^{1/2} f_\varepsilon(a) = \lim_{n\to\infty} u_{n+1} \cdots u_1 x^{1/2} f_\varepsilon(a) = \lim_{n\to\infty} u_{n+1} \cdots u_1 f_\varepsilon(a) = f_\varepsilon(a).$$

One also notices that the third part of the lemma holds from the proof above as we may replace a by $u_1 a u_1^*$ with $u_1 f_{\varepsilon/4}(a) u_1^* = f_{\varepsilon/4}(u_1 a u_1^*) \in \mathrm{Her}(b)$.

□

Corollary 2.12. *Let A be a C^*-algebra which has almost stable rank one, and $a \in (A \otimes \mathcal{K})_+$ (or $a \in A_+$) and $b \in A_+$. Then $a \lesssim b$ if and only if there is $x \in A \otimes \mathcal{K}$ (or $x \in A$) such that $x^*x = a$ and $xx^* \in \mathrm{Her}(b)$.*

Moreover, if $a_1, a_2, ..., a_n$ are mutually orthogonal elements in A_+ such that $a_i \sim a_1$ in $\mathrm{Cu}(A)$ for $i = 1, 2, ..., n$, and $a := \sum_{i=1}^n a_i \lesssim b$, then there is a hereditary C^*-subalgebra $A_1 \subset \mathrm{Her}(b)$ such that there is an isomorphism $\varphi : M_n(A_2) \to A_1$ where $A_2 = \mathrm{Her}(d)$ for some $d \in \mathrm{Her}(b)$ such that $\varphi^{-1}(d) = d$ and there is $z \in A$ such that $z^*z = a_1$ and $zz^* = d$.

Proof. The first part follows from Lemma 2.11. In fact, in the second part of Lemma 2.11, we choose $x = va^{1/2}$. Then $x^*x = a^{1/2}v^*va^{1/2} = a$ and $xx^* = vav^* \in \mathrm{Her}(b)$.

By Lemma 2.11, there is $v \in A^{**}$ such that $\bar{a} := vav^* \in \mathrm{Her}(b)$ and $vc, cv \in \mathrm{Her}(b)$ and $v^*vc = c$ for all $c \in \mathrm{Her}(a)$. Let $y_0 = va_1v^*$. Then, by the first part of this lemma, there is $z \in A$ such that $z^*z = a_1$ and $b_1 := zz^* \in \mathrm{Her}(y_1)$. Note that $b_1 \sim va_iv^*$ in $\mathrm{Her}(\bar{a}) \subset \mathrm{Her}(b)$, $i = 1, 2, ..., n$. Thus we have $x_i \in \mathrm{Her}(\bar{a})$ such that $x_i^*x_i = b_1$ and $x_ix_i^* \in \mathrm{Her}(va_iv^*)$, $i = 1, 2, ..., n$. Note that $x_1x_1^* = x_1^*x_1 = b_1$. Put $A_1 = \mathrm{Her}(\sum_{i=1}^n x_ix_i^*)$. One then checks that $A_1 = M_n(A_2)$, where $A_2 = \mathrm{Her}(b_1)$. The corollary follows.

□

We would like to end this section with the following folklore.

Lemma 2.13. *Let A be a C^*-algebra and $0 \le a \le b \le 1$ be elements in A. Then, for any $0 < \varepsilon < \varepsilon' < \|a\|$, there exists $z \in A$ such that*

$$(a-\varepsilon)_+ \lesssim (b-\varepsilon)_+, \ \ (a-\varepsilon')_+ \le z^*z \ \text{ and } \ zz^* \in \mathrm{Her}((b-\varepsilon)_+). \tag{e 2.17}$$

Proof. Choose $0 < \varepsilon < \varepsilon' < \varepsilon'' < \|a\|$ and define $g \in C_0((0,1])$ such that $g(t) = 1$ for $t \in [\varepsilon'', 1]$ and $(t-\varepsilon')_+ \le g(t) \le 1$ for $t \in (\varepsilon', \varepsilon'')$, $g(t) = 0$ if $t \in (0, \varepsilon')$. Then $(a-\varepsilon')_+ \le g(a)$ and

$$(\varepsilon')g(a) \le g(a)^{1/2}ag(a)^{1/2} \le g(a)^{1/2}bg(a)^{1/2}. \tag{e 2.18}$$

It follows that

$$g(a)^{1/2}((b-\varepsilon)_+)g(a)^{1/2} \ge g(a)^{1/2}(b-\varepsilon)g(a)^{1/2} \tag{e 2.19}$$
$$= g(a)^{1/2}bg(a)^{1/2} - \varepsilon g(a) \ge (\varepsilon' - \varepsilon)g(a). \tag{e 2.20}$$

Thus

$$(a-\varepsilon')_+ \le g(a) \le (1/(\varepsilon'-\varepsilon))g(a)^{1/2}(b-\varepsilon)_+g(a)^{1/2} \lesssim (b-\varepsilon)_+. \tag{e 2.21}$$

Since the above holds for any $0 < \varepsilon < \varepsilon'$, $(a-\varepsilon)_+ \lesssim (b-\varepsilon)_+$. Let $z = (1/(\varepsilon'-\varepsilon))^{1/2}(b-\varepsilon)_+^{1/2}g(a)^{1/2}$. Then

$$g(a) \le z^*z \ \text{ and } \ zz^* = (1/(\varepsilon'-\varepsilon))(b-\varepsilon)_+^{1/2}g(a)(b-\varepsilon)_+^{1/2} \in \mathrm{Her}((b-\varepsilon)_+). \tag{e 2.22}$$

□

3 Unitary groups

The main purpose of this section is to present a K_1-cancellation result for separable regular simple C^*-algebras.

Definition 3.1. Let A be a C^*-algebra. Denote by $\tilde{A}$ the C^*-algebra generated by A and $\mathbb{C} \cdot 1_{\tilde{A}}$, where $1_{\tilde{A}}$ is not in A. Denote by $\pi_{\mathbb{C}}^A : \tilde{A} \to \mathbb{C} \cdot 1_{\tilde{A}} = \mathbb{C}$ the quotient map. We also write $\pi_{\mathbb{C}}^A$ for the extension from $M_n(\tilde{A})$ to M_n.

Definition 3.2. Let A be a unital C^*-algebra. Denote by $U(A)$ the unitary group of A and by $U_0(A)$ the path connected component of $U(A)$ containing 1_A.

Proposition 3.3. *Let A be a C^*-algebra and $u \in U_0(M_n(\tilde{A}))$ be a unitary with the form $u = \alpha \cdot 1_{M_n(\tilde{A})} + a$ for some $\alpha \in \mathbb{T}$ and $a \in M_n(A)$. Then $u \in U_0(M_n(A)^\sim)$. In particular, if $\alpha = 1$, then $u = \exp(ib_1)\exp(ib_2)\cdots\exp(ib_m)$ for some $b_1, b_2, ..., b_m \in M_n(A)_{s.a.}$.*

Proof. Replacing u by $u\bar{\alpha}$, we may assume that $\pi_{\mathbb{C}}^A(u) = 1_n := 1_{M_n}$. Let $u = \exp(ih_1)\exp(ih_2)\cdots\exp(ih_k)$, where $h_j \in M_n(\tilde{A})_{s.a.}$. For each h_j, there is a scalar self-adjoint matrix $a_j \in M_n(\mathbb{C}\cdot 1_{\tilde{A}})$ such that $\pi_{\mathbb{C}}^A(h_j) = \pi_{\mathbb{C}}^A(a_j)$. Note that, since $\pi_{\mathbb{C}}^A(u) = 1_n$,

$$\exp(ia_1)\exp(ia_2)\cdots\exp(ia_k) = 1_n.$$

Define, for $t \in [0,1]$,

$$u(t) = \exp(ith_1)\exp(ith_2)\cdots\exp(ith_k)\exp(-ita_k)\exp(-ita_{k-1})\cdots\exp(-ita_1).$$

Then $u(1) = u(\exp(ia_1)\exp(ia_2)\cdots\exp(ia_k))^* = u$ and $u(0) = 1_n$. However,

$$\pi_{\mathbb{C}}^A(u((t))) = \exp(ita_1)\exp(ita_2)\cdots\exp(ita_k)\exp(-ita_k)\exp(-ita_{k-1})\cdots\exp(-ita_1) = 1_n.$$

Therefore $u(t) \in M_n(A)^\sim$ for all $t \in [0,1]$.

Suppose that $\alpha = 1$. Since now $u \in U_0(M_n(A)^\sim)$, $u = \exp(ih_1)\exp(ih_2)\cdots\exp(ih_m)$ for some $h_1, h_2, ..., h_m \in M_n(A)^\sim$. Let $\pi_{\mathbb{C}}^A(h_j) = \lambda_j \cdot 1_{M_n}$, where $\lambda_i \in \mathbb{T}$, $j = 1, 2, ..., m$. Then $\sum_{j=1}^m \lambda_j = 2k\pi$ for some integer k. Choose $b_j = h_j - \lambda_j\, (= h_j - \lambda_j 1_{M_n})$, $j = 1, 2, ..., m$. Then $b_j \in M_n(A)$. Note $\lambda_j \cdot 1_{M_n}$ is in the center of $M_n(A)^\sim$. Then

$$\exp(ib_1)\exp(ib_2)\cdots\exp(ib_m) = \exp(ih_1)\exp(ih_2)\cdots\exp(ih_m)\exp(i\sum_{j=1}^m -\lambda_j) = u.$$

□

Note, in the following statement, that the unital C^*-algebra $\tilde{A}$ is not divisible in any sense.

Lemma 3.4. *Let A be a finite regular simple C^*-algebra which has no nonzero projections, $u \in U(\tilde{A})$, and $a_1, a_2, ..., a_m \in A_{s.a.}$. Then, for any $a \in A_+ \setminus \{0\}$, any $\varepsilon > 0$, there is an integer $n_0 \geq 2$ such that, for any integer $n \geq n_0$, there is a hereditary C^*-subalgebra $B \subset A$, and a unitary $v \in \mathbb{C}\cdot 1_{\tilde{A}} + B$, $b_1, b_2, ..., b_m \in B$ such that $B = U^*(M_n(\mathrm{Her}((c-\eta)_+)))U$ for some unitary $U \in M_n(\tilde{A})$ and for some $0 < \eta < \|c\|$, where $c \in \mathrm{Her}(a)_+$ such that*

$$\|v - u\| < \varepsilon \ \text{ and } \ \|a_j - b_j\| < \varepsilon/2(m+1),\ 1 \leq j \leq m. \tag{e 3.1}$$

Moreover, we may assume that $4[c] \leq [a]$.

(Note that here we identify A with the first corner of $M_n(A)$.)

Proof. Fix a strictly positive element of e_A of A with $\|e_A\| = 1$. Write $u = \alpha\cdot 1_{\tilde{A}} + x$ for some $x \in A$ and $\alpha \in \mathbb{T}$. Let $1/2 > \varepsilon > 0$. Choose $1/2 > \delta > 0$ such that

$$\|f_\delta(e_A)xf_\delta(e_A) - x\| < \varepsilon/4 \ \text{ and } \ \|f_\delta(e_A)a_jf_\delta(e_A) - a_j\| < \varepsilon/2(m+1),\ j = 1, 2, ..., m. \tag{e 3.2}$$

Choose $b_j := f_\delta(e_A)a_jf_\delta(e_A)$, $1 \leq j \leq m$. Let $A_1 = \mathbb{C}\cdot 1_{\tilde{A}} + \mathrm{Her}(f_\delta(e_A))$. It is standard to find a unitary $v \in A_1$ such that

$$\|v - u\| < \varepsilon. \tag{e 3.3}$$

Let $D = \mathrm{Her}(f_{\delta/2}(e_A))$. Note that $D \subset \mathrm{Ped}(A)$. Choose $0 < \delta_0 < \delta/2$ such that $f_{\delta_0}(a) \neq 0$. Since both $f_{\delta_0}(a)$ and $f_{\delta/16}(e_A)$ are in $\mathrm{Ped}(A)$, there is an integer $k > 2$ such that

$$(k-1)[a] \geq (k-1)[f_{\delta_0}(a)] \geq [f_{\delta/16}(e_A)]. \tag{e 3.4}$$

Choose $n_0 = 4k$. Let $n \geq n_0$. Let $c_0 \in A \otimes \mathcal{K}$ with $0 \leq c_0 \leq 1$ such that $d_\tau(c_0) = (1/n)d_\tau(f_{\delta/8}(e_A))$ for all $\tau \in \widetilde{QT}(A)$. Thus

$$4d_\tau(c_0) < d_\tau(a) \text{ for all } \tau \in \widetilde{QT}(A) \setminus \{0\}. \tag{e 3.5}$$

It follows that $4[c_0] \leq [a]$ in $\mathrm{Cu}(A)$. Since A has almost stale rank one, by the first part of Lemma 2.11, there is $c \in \mathrm{Her}(a)_+$ such that $c \sim c_0$ and $d_\tau(c) = (1/n)\tau(f_{\delta/8}(e_A))$ for all $\tau \in \widetilde{QT}(A)$. Since $\mathrm{Cu}(A) = (V(A) \setminus \{0\}) \sqcup \mathrm{LAff}_+(\widetilde{QT}(A))$ and A has no non-zero projection,

$$4[c] \leq [a] \text{ and } [f_{\delta/8}(e_A)] = n[c]. \tag{e 3.6}$$

We now view A as a C^*-subalgebra of $M_n(A)$ (as the first corner of $M_n(A)$). Let

$$c_1 := \mathrm{diag}(\overbrace{c, c, ..., c}^{n}).$$

Then, by (e 3.6), $f_{\delta/2}(e_A) \ll f_{\delta/8}(e_A) \lesssim c_1$. Therefore there is $0 < \eta_0 < 1$ such that

$$f_{\delta/2}(e_A) \lesssim f_{\eta_0}(c_1). \tag{e 3.7}$$

Choose $0 < \eta < \eta_0/2$. Since A has almost stable rank one, from the last part of (e 3.6), by Lemma 2.10, there is a unitary $U_1 \in M_n(\tilde{A})$ such that

$$c_2 := U_1(c_1 - \eta)_+ U_1^* \in A. \tag{e 3.8}$$

By (e 3.7), since A has almost stable rank one, applying Lemma 2.10 again, there is a unitary $U_2 \in \tilde{A}$ such that

$$U_2^* f_\delta(e_A) U_2 \in \mathrm{Her}(c_2). \tag{e 3.9}$$

Put $c_3 = U_2 c_2 U_2^*$. Put $U = \mathrm{diag}(\overbrace{U_2, 1_{\tilde{U}}, ..., 1_{\tilde{U}}}^{n-1})U_1$. Then $f_\delta(e_A) \in \mathrm{Her}(c_3)$. Moreover, $B := \mathrm{Her}(c_3) = U^* M_n(\mathrm{Her}((c-\eta)_+))U^*$. Then $v \in \mathbb{C} \cdot 1_{\tilde{A}} + \mathrm{Her}(c_3)$.

□

Lemma 3.5. *Let A be a finite separable regular simple C^*-algebra and let $u \in \tilde{A}$ be a unitary. If* $\mathrm{diag}(u, 1) \in U_0(M_2(\tilde{A}))$, *then* $u \in U_0(\tilde{A})$.

Proof. Note that, if A has a nonzero projection, then, by Proposition 2.2, A has stable rank one. Then the lemma follows from Theorem 2.10 of [29]. So we now assume that A has no nonzero projection.

We may assume that $\pi_{\mathbb{C}}^A(\mathrm{diag}(u, 1)) = 1_2$. By the second part of Proposition 3.3, we may write $u = \exp(ib_1)\exp(ib_2)\cdots\exp(ib_m)$ for some $b_j \in M_2(A)_{s.a.}$, $1 \leq j \leq m$. Let $1/2 > \varepsilon > 0$. By virtue of Lemma 3.4, without loss of generality, we may assume that $u \in 1_{\tilde{A}} + B$ and there are $a_1, a_2, ..., a_m \in M_2(B)_{s.a.}$ such that

$$\|\mathrm{diag}(u, 1) - \exp(ia_1)\exp(ia_2)\cdots\exp(ia_m)\| < \varepsilon, \tag{e 3.10}$$

where $B = U^* M_n(\mathrm{Her}(c))U \subset A$ for some $c \in A_+$, $n \geq 4$, and where $U \in M_n(\tilde{A})$ (recall that we identify A with the first corner of $M_n(A)$). Put $C = U^*\mathrm{Her}(c)U$.

Write $u = 1_{\tilde{A}} + b$ for some $b \in B$. Let $u_1 := 1_{\tilde{B}} + b \in \tilde{B}$ and

$$v_1 := (1_{\tilde{B}} + \sum_{n=1}^{\infty} \frac{ia_1^n}{n!}) \cdot (1_{\tilde{B}} + \sum_{n=1}^{\infty} \frac{ia_2^n}{n!}) \cdot \cdots \cdot (1_{\tilde{B}} + \sum_{n=1}^{\infty} \frac{ia_m^n}{n!}). \tag{e 3.11}$$

Hence

$$\|\mathrm{diag}(u_1, 1_{\tilde{B}}) - v_1\| < \varepsilon. \tag{e 3.12}$$

Thus $\mathrm{diag}(u_1, 1_{\tilde{B}}) \in U_0(M_2(B)^\sim)$. Recall that A has almost stable rank one. Thus the set of invertible elements of $\tilde{C}$ is dense in $C = U^*\mathrm{Her}(c)U$, C has stable rank at most 2 (see the proof of Theorem 6.13 of [32]), by Theorem 2.10 of [29], the map from $U(M_n(\tilde{C}))/U_0(M_n(\tilde{C}))$ to $U(M_{2n}(\tilde{C}))/U_0(M_{2n}(\tilde{C}))$ is injective. It follows that $u_1 \in U_0(M_n(\tilde{C}))$. By Lemma 3.3, $u_1 \in U_0(M_n(C)^\sim) = U_0(\tilde{B})$. It follows that $u \in U_0(\tilde{A})$.

□

Theorem 3.6. *Let A be a separable finite regular simple C^*-algebra and let $u \in U(\tilde{A})$.*

(1) For any $a \in A_+ \setminus \{0\}$, there is a unitary $v \in \mathbb{C} \cdot 1_{\tilde{A}} + \mathrm{Her}(a)$ such that $uv^* \in U_0(\tilde{A})$.

(2) If $u = \alpha \cdot 1_{\tilde{A}} + x$ for some $\alpha \in \mathbb{T}$ and $x \in D$ for some hereditary C^*-subalgebra D of A and $u \in U_0(\tilde{A})$, then $v = \alpha \cdot 1_{\tilde{D}} + x \in U_0(\tilde{D})$.

Proof. If A has stable rank one, the theorem is well known and follows from the fact ([6]) that every nonzero (full) hereditary C^*-subalgebra D of A is stably isomorphic to A and the inclusion $\iota : D \to A$ induces an isomorphism on $K_1(D)$, and then apply Theorem 2.10 of [29].

We will prove the case that A is not assumed to have stable rank one. Therefore we assume A has no nonzero projection (see Proposition 2.2).

For (1), by Lemma 3.4, without loss of generality, we may assume $u = 1_{\tilde{A}} + b$ for some $b \in B$, where $B = U^*M_n(\mathrm{Her}((c-\eta)_+))U \subset A$ for some $0 < \eta < \|c\|$, and $c \in \mathrm{Her}(a)_+$, $n > 8$ and $4[c] \le [a]$, and where $U \in U(M_n(\tilde{A}))$.

Put $C = U^*\mathrm{Her}((c-\eta)_+)U$ and $u_1 := 1_{\tilde{B}} + b$. Since $GL(\tilde{C})$ is dense in C, by (the proof of) Theorem 6.13 of [32], C has stable rank at most 2. It follows from Proposition 5.3 of [29] that there exists a unitary $v_0 \in M_2(\tilde{C})$ such that $u_1 v_1^* \in U_0(M_n(\tilde{C}))$, where $v_1 := \mathrm{diag}(v_0, \overbrace{1_{\tilde{C}}, 1_{\tilde{C}}, ..., 1_{\tilde{C}}}^{n-2})$.

Let $w \in M_2(\mathbb{C} \cdot 1_{\tilde{C}})$ be the scalar matrix such that $\pi_{\mathbb{C}}^C(v_0) = \pi_{\mathbb{C}}^C(w)$. By replacing v_0 by $v_0 w^*$, we may assume that $\pi_{\mathbb{C}}^C(v_0) = 1_{M_2(\tilde{C})}$. Hence $v_0 \in M_2(C)^\sim$. Write $v_0 = 1_{M_2(\tilde{C})} + y$ for some $y \in M_2(C)$. It follows that $\pi_{\mathbb{C}}^B(u_1 v_1^*) = 1$. Then, by Lemma 3.3, $u_1 v_1^* \in U_0(\tilde{B})$. Let $v_2 := 1_{\tilde{A}} + y$. Then $uv_2^* \in U_0(\tilde{A})$. Since $4[c] \le [a]$ and A has almost stable rank one, by Lemma 3.2 of [16], there is a unitary $V \in \tilde{A}$ such that (note that $\eta > 0$)

$$V^*M_2(C)V \subset \mathrm{Her}(a). \tag{e 3.13}$$

Then $V^*yV \in \mathrm{Her}(a)$. Define $v = V^*v_2V$. Then v has the form described in the lemma. Put $W := V^*uv_2^*V$. Since $uv_2^* \in U_0(\tilde{A})$, one has

$$\begin{pmatrix} W & 0 \\ 0 & 1 \end{pmatrix} \in U_0(M_2(\tilde{A})). \tag{e 3.14}$$

Applying Lemma 3.5, one concludes $W \in U_0(\tilde{A})$. Thus

$$(V^*uV)v^* \in U_0(\tilde{A}). \tag{e 3.15}$$

There exists a continuous path of unitaries $\{H(t) : t \in [0,1]\} \subset U(M_2(\tilde{A}))$ such that

$$H(0) = \begin{pmatrix} V^*uV & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \text{ and } H(1) = \begin{pmatrix} u & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} uv^* & 0 \\ 0 & 1 \end{pmatrix}. \tag{e 3.16}$$

By (e 3.15), $H(0) \in U_0(M_2(\tilde{A}))$. Therefore $\mathrm{diag}(uv^*, 1) \in U_0(M_2(\tilde{A}))$. Applying Lemma 3.5 again, one obtains $uv^* \in U_0(\tilde{A})$.

To see part (2), we may assume that $\alpha = 1$. Let $\iota : D \to A$ be the inclusion map. Since D is a full hereditary C^*-algebra and A is separable, it follows that D is stably isomorphic to A and $\iota_{*1} : K_1(D) \to K_1(A)$ is an isomorphism (see, for example, Corollary 2.10 of [6]). Let $u_1 = 1_{\tilde{D}} + x$. Then $\iota_{*1}([u_1]) = [u]$ is zero in $K_1(A)$ from the assumption that $u \in U_0(\tilde{A})$. Thus $[u_1]$ is zero in $K_1(D)$. Therefore, for some integer $n \geq 1$,

$$\mathrm{diag}(u_1, \overbrace{1_{\tilde{D}}, ..., 1_{\tilde{D}}}^{2n+1}) \in U_0(M_{2n}(\tilde{D})). \tag{e 3.17}$$

Since D is a finite separable regular simple C^*-algebra, by repeatedly applying Lemma 3.5, we conclude that $u_1 \in U_0(\tilde{D})$.

□

Corollary 3.7. *Let A be a separable regular simple C^*-algebra. Then the map*

$$U(M_n(\tilde{A}))/U_0(M_n(\tilde{A})) \to U(M_{n+1}(\tilde{A}))/U_0(M_{n+1}(\tilde{A})) \tag{e 3.18}$$

is an isomorphism. In particular, $U(\tilde{A})/U_0(\tilde{A}) = K_1(A)$.

Proof. The finite case follows immediately from Theorem 3.6. Suppose that A is a purely infinite simple C^*-algebra. By the comment before Remark 3.1 of [7], eAe is extremally rich for any projection $e \in A$. Applying Proposition 5.4 of [7], one concludes that A is extremally rich. Since $\mathbb{C}$ has stable rank one, by Proposition 6.8 of [7], $\tilde{A}$ is extremally rich. By [43], A has real rank zero, and, hence, $\tilde{A}$ has real rank zero. By theorem 6.10 of [8], the corollary follows (when A is a purely infinite simple C^*-algebra). □

Corollary 3.8. *Let A be a separable simple $\mathcal{Z}$-stable C^*-algebra. Then the map*

$$U(M_n(\tilde{A}))/U_0(M_n(\tilde{A})) \to U(M_{n+1}(\tilde{A}))/U_0(M_{n+1}(\tilde{A})) \tag{e 3.19}$$

is an isomorphism. In particular, $U(\tilde{A})/U_0(\tilde{A}) = K_1(A)$.

4 Comparison in $\tilde{B}$

The main purpose of this section is to present Theorem 4.11 and Theorem 4.12.

4.1. Let A be a separable simple C^*-algebra and let $\widetilde{T}(A)$ be the cone of densely defined positive lower semi-continuous traces on A equipped with the topology of point-wise convergence on elements of the Pedersen ideal $\mathrm{Ped}(A)$ of A. By Proposition 3.4 of [37], $\widetilde{T}(A)$ has a Choquet simplex T_e as its base. Let f be a lower semicontinuous affine function on $\widetilde{T}(A)$ such that $f(t) > 0$ for all $t \in \widetilde{T}(A) \setminus \{0\}$. Then, a standard compactness argument shows that $\inf\{f(t) : t \in T_e\} > 0$. By I.1.4 of [1], together with a standard compactness argument, one obtains an increasing sequence $f_n \in \mathrm{Aff}_+(\widetilde{T}(A))$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ for all $t \in \widetilde{T}(A)$. In other words, $f \in \mathrm{LAff}_+(\widetilde{T}(A))$.

Now suppose that A is a finite separable regular simple C^*-algebra. It follows that $M_n(A)$ has almost stable rank one, for all $n \in \mathbb{N}$. Let us assume that every densely defined 2-quasitrace is a trace. Then $\mathrm{LAff}_+(\widetilde{QT}(A)) = \mathrm{LAff}_+(\widetilde{T}(A))$. Let $a \in \mathrm{Ped}(A)_+ \setminus \{0\}$. Then $C = \mathrm{Her}(a)$ is algebraically simple. Choose $f \in \mathrm{Aff}_+(\widetilde{T}(A)) \setminus \{0\}$ such that $f(\tau) < d_\tau(a)$ for all $\tau \in \widetilde{T}(A) \setminus \{0\}$. Then there is $c \in (A \otimes \mathcal{K})_+$ such that $d_\tau(c) = f(\tau)$ for all $\tau \in \widetilde{T}(A)$, and $c \lesssim a$. Since A has almost stable rank one, by 2.12, there exists $x \in A \otimes \mathcal{K}$ such that $xx^* = c$ and $b := x^*x \in C_+$. Note that $d_\tau(b) = f(\tau)$ for all $\tau \in \widetilde{T}(A)$. By Theorem 5.3 of [16], $\mathrm{Her}(b)$ has continuous scale. Note also that $\mathrm{Her}(b) \otimes \mathcal{K} \cong A \otimes \mathcal{K}$.

In the case that $QT(A) = T(A)$ and $T(A)$ is compact, the map $f \mapsto f|_{T(A)}$ is affine and continuous, and an order isomorphism from $\mathrm{LAff}_+(\tilde{T}(A))$ onto $\mathrm{LAff}_+(T(A))$ as $\tilde{T}(A)$ is a convex topological cone with the metrizable Choquet simplex $T(A)$ as its base (note $0 \in \mathrm{Aff}_+(T(A))$–see 2.7). Therefore, since A is regular (see 2.9), in this case, $\mathrm{Cu}(A) = (V(A) \setminus \{0\}) \sqcup \mathrm{LAff}_+(T(A))$.

4.2. Throughout this section, B is, unless otherwise stated, a finite separable stably projectionless simple C^*-algebra with continuous scale such that $M_n(B)$ is regular for each integer $n \geq 1$, and $QT(B) = T(B)$ (for example, B is an exact finite separable simple stably projectionless $\mathcal{Z}$-stable C^*-algebra with continuous scale – see 2.9).

Note that, by (the proof of) Theorem 6.13 of [32], B has stable rank at most two. Also, since B has continuous scale, $T(B)$ is compact (see Theorem 5.3 of [16]). We also have, as B is stably projectionless, $\mathrm{Cu}(B) = \mathrm{LAff}_+(T(B))$.

If $a \in (\tilde{B} \otimes \mathcal{K})_+ \setminus \{0\}$, $\hat{a}(\tau) := \tau(a)$ for all $\tau \in T(B)$ is a function in $\mathrm{LAff}_+(T(B))$ (or for all $\tau \in T(\tilde{B})$ as a function in $\mathrm{LAff}_+(\tilde{B})$) and $\widehat{[a]}(\tau) := d_\tau(a)$ for all $\tau \in T(B)$ is a function $\mathrm{LAff}_+(T(B))$ (or for $\tau \in T(\tilde{B})$ as a function in $\mathrm{LAff}_+(T(\tilde{B}))$). Note that $\widehat{[a]}$ is a lower semicontinuous affine functions in $\mathrm{LAff}_+(T(B))$ with values in $(0, \infty]$.

Note that, if $a, b \in (B \otimes \mathcal{K})_+$ and $d_\tau(a) \leq d_\tau(b)$ for all $\tau \in T(B)$, then $a \lesssim b$ (recall that B is stably projectionless). In particular, B has strict comparison for positive elements.

Moreover, if $a, b \in (B \otimes \mathcal{K})_+$ and $[a] \leq [b]$ in $\mathrm{Cu}^\sim(B)$, then, as B has stable rank at most 2, by Corollary 4.10 of [32],

$$[a] + 2[1_{\tilde{B}}] \leq [b] + 2[1_{\tilde{B}}] \ \text{ in } \mathrm{Cu}(\tilde{B}). \tag{e 4.1}$$

It follows that $d_\tau(a) \leq d_\tau(b)$ for all $\tau \in T(B)$. Therefore $a \lesssim b$, or $[a] \leq [b]$ in $\mathrm{Cu}(B)$. This also implies that $\mathrm{Cu}(B)$ is orderly embedded into $\mathrm{Cu}^\sim(B)$.

These facts will be repeatedly used.

Note that $\tilde{B}$ is unital. Suppose that $B \neq \mathrm{Ped}(B)$. Let $a = d + b$, where $d \in M_r(\mathbb{C} \cdot 1_{\tilde{B}})_+ \setminus \{0\}$ and $b \in \mathrm{Ped}(B)_+$. Then $\tau(a) = \infty$ for those $\tau \in \tilde{T}(B)$ which is not bounded (see the last part of 4.9 of [15]). If B is stable, then $\tau(a) = \infty$ for all $\tau \in \tilde{T}(B)$. Therefore, it is more than convenient to consider a hereditary C^*-subalgebra of B which has continuous scale (see 4.1).

Definition 4.3. Let A be a unital C^*-algebra with stable rank at most m ($m \geq 1$). Denote by $\mathrm{Cu}(A)^{\circeq}$ the set of equivalence classes of elements in $\mathrm{Cu}(A)$ with the following equivalence relation: $x \circeq y$ if and only if $x + m[1_A] = y + m[1_A]$ in $\mathrm{Cu}(A)$. The the map $x \to (x, 0)$ gives an order embedding from $\mathrm{Cu}(A)^{\circeq}$ to $\mathrm{Cu}^\sim(A)$ (see 3.1 of [30], Subsection 4.2 and Corollary 4.10 of [32]). So, in this unital case, we may view $\mathrm{Cu}(A)^{\circeq} \subset \mathrm{Cu}^\sim(A)$.

Let B be a non-unital stably finite C^*-algebra with continuous scale and let $\tau_{\mathbb{C}}$ be the tracial state of $\tilde{B}$ that vanishes on B. Define

$$\mathrm{LAff}_+(T(\tilde{B}))^\diamond = \{f \in \mathrm{LAff}_+(T(\tilde{B})) : f(\tau_{\mathbb{C}}) \in \{0\} \cup \mathbb{N} \cup \{\infty\}\}. \tag{e 4.2}$$

Lemma 4.4 (Theorem A.6 of [17] and Theorem 6.11 of [32])**.** *Let B be in 4.2. Then* $\mathrm{Cu}(\tilde{B})^{\circeq} = (K_0(\tilde{B})_+ \setminus \{0\}) \sqcup \mathrm{LAff}_+(T(\tilde{B}))^\diamond$ *(see lines above Theorem 6.11 of [32] –also at the end of 2.8).*

Proof. By the assumption, applying Theorem 6.11 of [32], one obtains $\mathrm{Cu}^\sim(B) = K_0(B) \sqcup \mathrm{LAff}_+^\sim(T(B))$ (see also I.1.4 of [1] and the first part of 4.1). Note that, as in the proof of Theorem 6.13 of [32], B has stable rank at most 2. So, the definition of $\mathrm{Cu}^\sim(B)$ in [32] coincides with that in [30] (see subsection 4.2 of [32]).

Let $x, y \in \mathrm{Cu}(\tilde{B})$ which are not represented by projections and are represented by elements $a, b \in (\tilde{B} \otimes \mathcal{K})_+$ such that $[\pi_{\mathbb{C}}^B(a)] = n[1_{\tilde{B}}]$ and $[\pi_{\mathbb{C}}^B(b)] = m[1_{\tilde{B}}]$ for some integers $n, m \geq 0$, respectively. Suppose that $d_\tau(a) = d_\tau(b)$ for all $\tau \in T(\tilde{B})$. We will show that $x \mathrel{\mathring{=}} y$. Let $\tau_{\mathbb{C}}$ be the tracial state of $T(\tilde{B})$ which vanishes on B. The condition $d_{\tau_{\mathbb{C}}}(a) = d_{\tau_{\mathbb{C}}}(b)$ implies that $n = m$. It then follows from Theorem 6.11 of [32] that there exists k $(= 2)$ such that (in $\mathrm{Cu}(\tilde{B})$)

$$[a] + n[1_{\tilde{B}}] + k[1_{\tilde{B}}] = [b] + n[1_{\tilde{B}}] + k[1_{\tilde{B}}]. \tag{e 4.3}$$

Thus $x \mathrel{\mathring{=}} y$ (see also Corollary 4.10 of [32]).

Now consider the case that $[\pi_{\mathbb{C}}^B(a)] = \infty = [\pi_{\mathbb{C}}^B(b)]$. Then, for any $1 > \varepsilon > 0$, $f_\varepsilon(a) \in \mathrm{Ped}(B \otimes \mathcal{K})$. Hence $[\pi_{\mathbb{C}}^B(f_\varepsilon(a))] < \infty$. Also, there is $0 < \eta < \varepsilon$, as $[a]$ is not represented by a projection and B is simple,

$$d_\tau(f_\varepsilon(a)) < \tau(f_\eta(a)) \ \text{ for all } \ \tau \in T(B). \tag{e 4.4}$$

For $\pi_{\mathbb{C}}^B(b)$, since 0 is the only non-isolated point of the spectrum of $\pi_{\mathbb{C}}^B(b)$, one may find $g \in C_0((0, \|b\|])_+$ such that $[\pi_{\mathbb{C}}^B(g(b))] = [\pi_{\mathbb{C}}^B(f_\varepsilon(a))]$. Let $m := [\pi_{\mathbb{C}}^B(f_\varepsilon(a))] < \infty$. Note that $T(B)$ is compact. One then can find $0 < \delta < \eta/2$ such that $[\pi_{\mathbb{C}}^B(f_\delta(b))] = [\pi_{\mathbb{C}}^B(f_\eta(a))]$ and

$$d_\tau(f_\varepsilon(a)) < \tau(f_\eta(a)) < d_\tau(f_\delta(b)) \ \text{ for all } \ \tau \in T(B). \tag{e 4.5}$$

Consider $C := \overline{f_{\delta/4}(b)(B \otimes \mathcal{K})f_{\delta/4}(b)}$ and let $\{e_n\}$ be an approximate identity for $B \otimes \mathcal{K}$. Then

$$\tau(f_{\delta/4}(b)^{1/2} e_n f_{\delta/4}(b)^{1/2}) \nearrow \tau(f_{\delta/4}(b)) \ \text{ for all } \ \tau \in T(B). \tag{e 4.6}$$

It follows that (recall $T(B)$ is compact) there is $n_0 \geq 1$ such that, for all $n \geq n_0$,

$$\tau(f_{\delta/4}(b)^{1/2} e_n f_{\delta/4}(b)^{1/2}) > \tau(f_{\delta/2}(b)) \geq d_\tau(f_\delta(b)) > d_\tau(f_\varepsilon(a)) \ \text{ for all } \ \tau \in T(B). \tag{e 4.7}$$

Choose $b' = f_{\delta/4}(b)^{1/2} e_{n_0+1} f_{\delta/4}(b)^{1/2} + g(b)$. Then $[b'] \leq [b]$ in $\mathrm{Cu}(\tilde{B})$. We also have $[\pi_{\mathbb{C}}^B(b')] = [\pi_{\mathbb{C}}^B(g(b)] = [\pi_{\mathbb{C}}^B(f_\varepsilon(a))] = m$. It follows from (e 4.7) that

$$d_\tau(b') > d_\tau(f_\varepsilon(a)) \ \text{ for all } \ \tau \in T(B). \tag{e 4.8}$$

It follows that

$$d_\tau(b') - m \geq d_\tau(f_\varepsilon(a)) - m \ \text{ for all } \ \tau \in T(B). \tag{e 4.9}$$

It follows from Theorem 6.11 of [32] that in $\mathrm{Cu}^\sim(B)$,

$$(b', m) \geq (f_\varepsilon(a), m) \tag{e 4.10}$$

which means that, for some integer $k \geq 1$, in $\mathrm{Cu}(\tilde{B})$,

$$[b] + m[1_{\tilde{B}}] + k[1_{\tilde{B}}] \geq [b'] + m[1_{\tilde{B}}] + k[1_{\tilde{B}}] \geq [f_\varepsilon(a)] + m[1_{\tilde{B}}] + k[1_{\tilde{B}}]. \tag{e 4.11}$$

Since B has stable rank at most two, by Corollary 4.10 of [32], $[b] + 2[1_{\tilde{B}}] \geq [f_\varepsilon(a)] + 2[1_{\tilde{B}}]$. Since the above holds for all $0 < \varepsilon < 1$, one concludes that

$$[b] + 2[1_{\tilde{B}}] \geq [a] + 2[1_{\tilde{B}}]. \tag{e 4.12}$$

The same argument also shows that

$$[a] + 2[1_{\tilde{B}}] \geq [b] + 2[1_{\tilde{B}}]. \tag{e 4.13}$$

It follows that $[a] \mathrel{\mathring{=}} [b]$. This shows that the map from $\mathrm{Cu}(\tilde{B})^{\mathring{=}}$ to $(K_0(\tilde{B})_+ \setminus \{0\}) \sqcup \mathrm{LAff}_+(T(\tilde{B}))^\diamond$ is an order embedding.

The map is surjective follows from the first part of Theorem A.6 (and Def. A.5) of [17].

$\square$

Lemma 4.5. *Let A be a C^*-algebra which has almost stable rank one. Suppose that $a \in M_r(A)$ and $b \in M_r(\tilde{A})$ (for some $r \ge 1$). Then $\mathrm{dist}(x, LG(M_{2r}(\tilde{A})) = 0$, where $LG(M_{2r}(\tilde{A}))$ is the set of invertible elements in $M_{2r}(\tilde{A})$ and $x = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$.*

Proof. For any $\varepsilon > 0$, by Proposition 2.4, there is an invertible element $y \in M_r(\tilde{A})$ with the inverse y^{-1} such that $\|a - y\| < \varepsilon$. Put

$$z := \begin{pmatrix} y & b \\ 0 & \varepsilon \end{pmatrix} \text{ and } w := \begin{pmatrix} y^{-1} & 0 \\ 0 & 1/\varepsilon \end{pmatrix}. \tag{e 4.14}$$

Then $\|x - z\| < \varepsilon$ and w is invertible and

$$zw = \begin{pmatrix} 1 & b/\varepsilon \\ 0 & 1 \end{pmatrix}. \tag{e 4.15}$$

Since $\begin{pmatrix} 0 & b/\varepsilon \\ 0 & 0 \end{pmatrix}$ is a nilpotent, zw is invertible. As w is invertible, z is invertible.

□

Lemma 4.6. *Let B be as in 4.2. Suppose that $a, b \in M_r(B)_+$ and $c, d \in M_r(\tilde{B})_+$ such that $a \lesssim b$ (in $M_r(B)$) and $c \lesssim d$ (in $M_r(\tilde{B})$) for some integer $r \ge 1$. Suppose also that $b \perp d$.*

Then, for any $\eta > 0$ and $\varepsilon > 0$, there is a unitary $U \in M_{2r}(\tilde{B})$, $\delta > 0$, and $h \in \mathrm{Her}(f_\delta(b+d))_+$ with $\|h\| \le 1$ such that

$$\|U^* f_\eta(a + c)U - h\| < \varepsilon. \tag{e 4.16}$$

(We identify $M_r(\tilde{B})$ with the hereditary C^*-subalgebra $\{\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \in M_{2r}(\tilde{B}) : b \in M_r(\tilde{B})\}$.)

Proof. Fix $1 > \eta > 0$. There are $x_1 \in M_r(B)$ and $x_2 \in M_r(\tilde{B})$ such that (see 2.12)

$$x_1^* x_1 = a, \;\; x_1 x_1^* \in \mathrm{Her}(b), \;\; \|x_2^* x_2 - c\| < \eta/4, \;\text{ and }\; x_2 x_2^* \in \mathrm{Her}(d). \tag{e 4.17}$$

Put $x := \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix}$. Then $x^*x = \begin{pmatrix} x_1^*x_1 + x_2^*x_2 & 0 \\ 0 & 0 \end{pmatrix}$. By Proposition 2.2 of [33], there is $r \in M_r(\tilde{B})$ such that $r^*(x_1^*x_1 + x_2^*x_2)r = f_{\eta/2}(a+c)$. Let $y = \begin{pmatrix} x_1 r & 0 \\ x_2 r & 0 \end{pmatrix}$. Then $y^*y = \begin{pmatrix} f_{\eta/2}(a+c) & 0 \\ 0 & 0 \end{pmatrix}$.

Let $y = v|y|$ be the polar decomposition of y in $M_{2r}(\tilde{B})^{**}$. By applying Lemma 4.5 above and Theorem 5 of [27], there is, for any $\sigma > 0$, a unitary $W \in M_{2r}(\tilde{B})$ such that

$$W f_\sigma(|y|) = v f_\sigma(|y|). \tag{e 4.18}$$

We choose a sufficiently small σ so that

$$W f_\eta(a + c)^{1/2} = v f_\eta(a + c)^{1/2}. \tag{e 4.19}$$

Then

$$W f_\eta(a + c) W^* = v f_\eta(a + c) v^* \le f_{\eta/4}(yy^*). \tag{e 4.20}$$

Note that (see I.1.11 of [2])

$$yy^* = \begin{pmatrix} x_1 r r^* x_1^* & x_1 r r^* x_2^* \\ x_2 r r^* x_1^* & x_2 r r^* x_2^* \end{pmatrix} \le 2 \begin{pmatrix} x_1 r r^* x_1^* & 0 \\ 0 & x_2 r r^* x_2^* \end{pmatrix}. \tag{e 4.21}$$

Thus $yy^* \in \mathrm{Her}(f)$, where $f := \begin{pmatrix} b^{1/2} & 0 \\ 0 & d^{1/2} \end{pmatrix}$.

Put $z := \begin{pmatrix} b^{1/2} & d^{1/2} \\ 0 & 0 \end{pmatrix}$. Then (recall $b \perp d$)

$$zz^* = b + d \ \text{ and } \ z^*z = \begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix}. \tag{e 4.22}$$

For any $1 > \varepsilon > 0$, choose $0 < \delta < 1/2$ such that

$$\|f_\delta(|z|)Wf_\eta(a+c)W^*f_\delta(|z|) - Wf_\eta(a+c)W^*\| < \varepsilon. \tag{e 4.23}$$

Let $z = u|z|$ be the polar decomposition of z in $M_{2r}(\tilde{B})^{**}$. By Lemma 4.5 above and Theorem 5 of [27] again, there is a unitary $W_1 \in M_{2r}(\tilde{B})$ such that $W_1 f_\delta(|z|) = uf_\delta(|z|)$. It follows that

$$W_1 f_\delta(|z|)W_1^* = f_\delta(b+d). \tag{e 4.24}$$

Let $h = W_1 f_\delta(|z|)Wf_\eta(a+c)W^*f_\delta(|z|)W_1^* \in \mathrm{Her}(f_\delta(b+d))_+$. Then $\|h\| \le 1$ and

$$\|W_1 W f_\eta(a+c)W^*W_1 - h\| < \varepsilon.$$

$\square$

Definition 4.7 (A.1 of [17])**.** Let B be a separable C^*-algebra with compact $T(B) \neq \emptyset$. Let $a \in M_n(\tilde{B})_+$ define

$$\omega([a]) = \inf\{\sup\{d_\tau(a) - \tau(c) : \tau \in T(B)\} : 0 \le c \le 1 \ \text{ and } \ c \in \overline{aM_n(\tilde{B})a}\}. \tag{e 4.25}$$

Note that $\widehat{[a]}$ is continuous on $T(B)$ if and only if $\omega([a]) = 0$, and if $a \sim b$, then $\omega([a]) = \omega([b])$. (see A.1 of [17]). If B has continuous scale and $p \in M_n(\tilde{B})$ is a projection, then $\hat{p}$ and $\widehat{1_{M_n(\tilde{B})} - p}$ are both lower semicontinuous. Thus both are continuous.

Lemma 4.8. *Let B be a nonunital simple C^*-algebra and $a \in M_n(\tilde{B})_+$ with $0 \le a \le 1$ such that 0 is not an isolated point of $\mathrm{sp}(a)$. Then, for any $1/2 > \delta_0 > 0$, there exists $0 < \delta < \delta_0$ such that there is an element $b \in \mathrm{Her}(f_\delta(a))_+ \setminus \{0\}$ such that $b \perp f_{\delta_0}(a)$ and there is a nonzero element $c \in M_n(B)_+ \cap \mathrm{Her}(b)_+$.*

Moreover, if $T(B)$ is a nonempty compact set, then

$$\inf\{\tau(c) : \tau \in T(B)\} > 0. \tag{e 4.26}$$

Proof. The existence of b follows from the spectral theory immediately. For the existence of c, note, since $b \neq 0$, $\overline{bM_n(B)b} \neq \{0\}$. Choose $c \in \overline{bM_n(B)b}_+ \setminus \{0\}$. By the simplicity of $M_n(B)$, $\tau(c) > 0$ for all $\tau \in T(B)$. Since we also assume that $T(B)$ is compact, inequality (e 4.26) holds. $\square$

Lemma 4.9 (Compare Lemma A.3 of [17])**.** *Let B be as in 4.2 and $a, b \in M_r(\tilde{B})_+$ (where $r \ge 1$ is an integer). Suppose that $\pi_{\mathbb{C}}^B(a) \lesssim \pi_{\mathbb{C}}^B(b)$ and*

$$d_\tau(a) + 4\omega([b]) < d_\tau(b) \ \text{ for all } \ \tau \in T(B). \tag{e 4.27}$$

Then, for any $1 > \eta > 0$, there exists a sequence of unitaries $U_n \in M_{2r}(\tilde{B})$ and a sequence of elements $h_n \in \mathrm{Her}(b)_+$ with $\|h_n\| \le 1$ such that

$$\lim_{n\to\infty} \|U_n^* f_\eta(a)U_n - h_n\| = 0. \tag{e 4.28}$$

In particular, $a \lesssim b$.

Proof. Let us assume that $0 \le a,\, b \le 1$. It suffices to prove that (e 4.28) holds for $f_{\eta_1}(a)$ in place of a for any $0 < \eta_1 < 1$. If $[a]$ is represented by a projection, then $d_\tau(a)$ is continuous. So

$$\inf\{d_\tau(b) - d_\tau(a) : \tau \in T(B)\} > 4\omega([b]). \tag{e 4.29}$$

Otherwise, for any fixed $0 < \eta_1 < 1/2$, there exist $\eta_1 > \eta_1/2 > \eta_2 > \eta_2/2 > \eta_3 > 0$ such that

$$d_\tau((a-\eta_1)_+) < \tau(f_{\eta_2}(a)) < d_\tau((a-\eta_3)_+) < d_\tau(a) \;\text{ for all }\; \tau \in T(B). \tag{e 4.30}$$

Then

$$\inf\{d_\tau(b) - d_\tau(f_{\eta_1}(a)) : \tau \in T(B)\} > 4\omega([b]). \tag{e 4.31}$$

Thus, in both cases, we may assume, without loss of generality (replacing a by $f_{\eta_1}(a)$) that

$$\inf\{d_\tau(b) : \tau \in T(B)\} > d = \inf\{d_\tau(b) - d_\tau(a) : \tau \in T(B)\} > 4\omega(b). \tag{e 4.32}$$

By applying Lemma A.2 of [17], one obtains non-zero elements $b_0 \in M_r(B)_+$ and $b_1, b' \in M_r(\tilde{B})_+$ with $b_0 \perp b_1$ such that

$$b_0 + b_1 \le b',\;\; [b'] = [b],\;\; \pi_{\mathbb{C}}^B(b_1) = \pi_{\mathbb{C}}^B(b'), \tag{e 4.33}$$

$$2\omega([b]) < d_\tau(b_0) < d/2,\;\; d_\tau(b_1) > d_\tau(b) - d/2 \;\text{ for all }\; \tau \in T(B), \tag{e 4.34}$$

and, for any $c_n' \in M_r(B)_+$ with $c_n' \in \overline{b_1 M_r(\widetilde{B}) b_1}$ and $d_\tau(c_n') \nearrow d_\tau(b_1)$ on $T(B)$, there exists $n_0 \ge 1$ such that

$$d_\tau(b_1) - d_\tau(c_n') < \omega([b]) + (1/64)\inf\{\tau(b_0) : \tau \in T(B)\} \;\text{ for all }\; \tau \in T(B). \tag{e 4.35}$$

In fact, the proof of Lemma A.2 of [17] states that $b' = g_{1,\eta_1}(b)$ for some strictly positive function g_{1,η_1} on $(0, \|b\|]$ as in the proof of Lemma A.2 of [17] (we replace a by b and a' by b'). Recall from A.1 of [17] that $\omega([b']) = \omega([b])$. Moreover, $[\pi_{\mathbb{C}}^B(b_1)] = [\pi_{\mathbb{C}}^B(b)]$. Replacing b by b', without loss of generality, we may assume that $b_0 + b_1 \le b$. We may also assume $0 \le b_0 + b_1 \le b \le 1$. Note, for any integer $m \ge 1$, that $b_0 + b_1^{1/m} \le b_0^{1/m} + b_1^{1/m} = (b_0 + b_1)^{1/m} \le b^{1/m}$. By choosing large m, we may assume that $\pi_{\mathbb{C}}^B(b_1^{1/m}) = \pi_{\mathbb{C}}^B(b^{1/m}) = p_1$ is a projection. Replacing b_1 by $b_1^{1/m}$ and b by $b^{1/m}$, we may further assume that $\pi_{\mathbb{C}}^B(b_1) = \pi_{\mathbb{C}}^B(b) = p_1$. Similarly, we may assume that $\pi_{\mathbb{C}}^B(a) := p_2$ is also a projection. Since $\pi_{\mathbb{C}}^B(a) \lesssim \pi_{\mathbb{C}}^B(b)$, there is a scalar matrix $U_0 \in M_r(\mathbb{C} \cdot 1_{\tilde{B}})$ such that $\pi_{\mathbb{C}}^B(U_0^* a U_0) \le p_1$. Hence we may also assume that $p_2 \le p_1$.

We may further assume that there are integers $m_2 \le m_1$ such that

$$p_i = \mathrm{diag}(\overbrace{1, 1, ..., 1}^{m_i}, 0, \cdots, 0),\;\; i = 1, 2. \tag{e 4.36}$$

Let $P_i = \mathrm{diag}(\overbrace{1_{\tilde{B}}, 1_{\tilde{B}}, ..., 1_{\tilde{B}}}^{m_i}, 0, \cdots 0)$, $i = 1, 2$.

Put $d_0 = \inf\{\tau(b_0) : \tau \in T(B)\}$. Note that the above holds for the case that $\omega([b]) = 0$. Note that $(b_1 - 1/n)_+ \le b_1$ and $d_\tau((b_1 - 1/n)_+) \nearrow d_\tau(b_1)$, so by (e 4.35), for some $\delta_1 > 0$,

$$d_\tau(b_1) - d_\tau(f_\delta(b_1)) < \omega([b]) + d_0/64 \;\text{ for all }\; \tau \in T(B) \tag{e 4.37}$$

and all $0 < \delta < \delta_1$. We also assume $\pi_{\mathbb{C}}^B(f_\delta(b_1)) = p_1$ $(0 < \delta \le \delta_1)$. Let $\{e_n\}$ be an approximate identity for B such that $e_n e_{n+1} = e_{n+1} e_n = e_n$, $n = 1, 2, ...$ Put

$$E_n = \mathrm{diag}(e_n, e_n, ..., e_n) \in M_r(B),\;\; n = 1, 2, ... \tag{e 4.38}$$

Then $\{E_n\}$ is an approximate identity for $M_r(B)$, and for all i and n,

$$E_nP_i = P_iE_n \text{ and } E_n(1-E_k) = 0 = (1-E_k)E_n, \text{ if } \; k \ge n+1. \qquad (e\,4.39)$$

We have $b_1^{1/2}E_nb_1^{1/2} \nearrow b_1$ (in the strict topology). Let $c_n = E_n^{1/2}b_1E_n^{1/2}$, $n = 1, 2, ...$ It follows that $d_\tau(c_n) \nearrow d_\tau(b_1)$ on $T(B)$. By the construction of b_1, there exists $n_0 \ge 1$ such that

$$d_\tau(b_1) - d_\tau(b_1^{1/2}E_nb_1^{1/2}) = d_\tau(b_1) - d_\tau(c_n) < \omega([b]) + d_0/64 \qquad (e\,4.40)$$

for all $\tau \in T(B)$ and for all $n \ge n_0$.

One then computes, by (e 4.40), (e 4.34) and (e 4.32), that, for $n \ge n_0$, for all $\tau \in T(B)$,

$$\begin{aligned} d_\tau(c_n) &> d_\tau(b_1) - \omega([b]) - d_0/64 > d_\tau(b) - d/2 - \omega([b]) - d_0/64 \\ &> d_\tau(a) + d/2 - \omega([b]) - d_0/64 > d_\tau(a) + d/4 - d_0/64 > d_\tau(a). \end{aligned} \qquad (e\,4.41)$$

Since $0 \le a \le 1$ and $\pi_{\mathbb{C}}^B(a) = \pi_{\mathbb{C}}^B(P_2)$, for any $0 < \eta < 1/2$, $\pi_{\mathbb{C}}^B(f_{\eta/2}(a)) = \pi_{\mathbb{C}}^B(a) = p_2$. Put $a_k = E_kf_{\eta/2}(a)E_k$, $k = 1, 2, ...$ Then, by (e 4.41), $a_k \lesssim c_n$ for any $k \ge 1$ and $n \ge n_0$, as B has the strict comparison.

On the other hand, since $\pi_{\mathbb{C}}^B(f_{\eta/2}(a)) = \pi_{\mathbb{C}}^B(a) = \pi_{\mathbb{C}}^B(P_2)$ and $\pi_{\mathbb{C}}^B(b_1) = \pi_{\mathbb{C}}^B(P_1)$,

$$b_1 = P_1 + b_{00}, \text{ and } f_{\eta/2}(a) = P_2 + a_{00}$$

for some $b_{00}, a_{00} \in M_r(B)_{s.a.}$. For any $\varepsilon > 0$, there is $k_{00} \ge 1$ such that, if $k \ge k_{00}$,

$$(1-E_k)b_1 \approx_\varepsilon (1-E_k)P_1 \text{ and } E_k^{1/2}b_{00} \approx_\varepsilon b_{00} \approx_\varepsilon b_{00}E_k^{1/2} \approx_\varepsilon E_k^{1/2}b_{00}E_k^{1/2}$$

Thus, by also (e 4.39), $E_k(P_1 + b_{00}) = E_k^{1/2}P_1E_k^{1/2} + E_kb_{00} \approx_{3\varepsilon} E_k^{1/2}b_1E_k^{1/2}$. Therefore, (with a similar consideration for $P_2 + a_{00}$)

$$\lim_{k\to\infty} \|(E_k^{1/2}b_1E_k^{1/2} + (1-E_k)^{1/2}P_1(1-E_k)^{1/2}) - b_1\| = 0 \text{ and} \qquad (e\,4.42)$$

$$\lim_{k\to\infty} \|(E_k^{1/2}f_{\eta/2}(a)E_k^{1/2} + (1-E_k)^{1/2}P_2(1-E_k)^{1/2}) - f_{\eta/2}(a)\| = 0. \qquad (e\,4.43)$$

Put $x_k := E_k^{1/2}f_{\eta/2}(a)E_k^{1/2} + (1-E_k)^{1/2}P_2(1-E_k)^{1/2}$ and $y_k := E_k^{1/2}b_1E_k^{1/2} + (1-E_k)^{1/2}P_1(1-E_k)^{1/2}$, $k = 1, 2, ...$ Since $y_n \to b_1$, we may also assume (by Proposition 2.2 of [33]) that, for all $n \ge n_0$,

$$f_{\delta_1/8}(y_n) \lesssim b_1. \qquad (e\,4.44)$$

Since, for any fixed $\delta_0 > 0$,

$$\lim_{k\to\infty} \|f_{\delta_0}(y_k) - f_{\delta_0}(b_1)\| = 0, \qquad (e\,4.45)$$

we may assume, without loss of generality, for all $k \ge 1$, $\pi_{\mathbb{C}}^B(f_{\delta_1/2}(y_k)) = p_1 = \pi_{\mathbb{C}}^B(f_{\delta_1/2}(b_1))$ and

$$\tau(f_{\delta_1/2}(y_k)) \ge \tau(f_{\delta_1/2}(b_1)) - d_0/64 \text{ for all } \tau \in T(B). \qquad (e\,4.46)$$

It follows by (e 4.37) (with $\delta = \delta_1/2$) that

$$\tau(f_{\delta_1/2}(y_k)) > d_\tau(b_1) - \omega([b]) - 3d_0/64 \text{ for all } \tau \in T(B). \qquad (e\,4.47)$$

Since $M_r(B)$ has continuous scale, there is $k_0 \ge n_0$ such that

$$d_\tau(1-E_n) \le \tau(1-E_{n-1}) < d_0/64 \text{ for all } \tau \in T(B) \text{ and for all } n \ge k_0. \qquad (e\,4.48)$$

It follows that, for $k \geq k_0$,

$$\tau(f_{\delta_1/2}(y_k)) \leq d_\tau(y_k) \leq d_\tau(c_k) + d_0/64 \tag{e 4.49}$$
$$= d_\tau(b_1^{1/2}E_k b_1^{1/2}) + d_0/64 \leq d_\tau(b_1) + d_0/64 \ \text{ for all } \ \tau \in T(B). \tag{e 4.50}$$

Let $g_{\delta_1} \in C_0((0,1])_+$ with $1 \geq g(t) > 0$ for all $t \in (0, \delta_1/4)$, $g_{\delta_1}(t) \geq t$ for $t \in (0, \delta_1/16)$, $g_{\delta_1}(t) = 1$ for $t \in (\delta_1/16, \delta_1/8)$ and $g_{\delta_1}(t) = 0$ if $t \geq \delta_1/4$.

Since $g_{\delta_1}(y_k)f_{\delta_1/2}(y_k) = 0$, by (e 4.49), we conclude that, for $k \geq k_0$,

$$d_\tau(g_{\delta_1}(y_k)) + \tau(f_{\delta_1/2}(y_k)) \leq d_\tau(y_k) \leq d_\tau(b_1) + d_0/64 \ \text{ for all } \ \tau \in T(B). \tag{e 4.51}$$

Then, by (e 4.47) and (e 4.34), for all $k \geq k_0$,

$$d_\tau(g_{\delta_1}(y_k)) \ \leq \ (d_\tau(b_1) - \tau(f_{\delta_1/2}(y_k))) + d_0/64 \tag{e 4.52}$$
$$\leq \ \omega([b]) + 3d_0/64 + d_0/64 = \omega([b]) + d_0/16 < d_0 \tag{e 4.53}$$

for all $\tau \in T(B)$. Moreover, since $\pi_{\mathbb{C}}^B(y_k) = p_1 = \pi_{\mathbb{C}}^B(f_{\delta_1/2}(y_k))$ for all k,

$$g_{\delta_1}(y_k) \in M_r(B). \tag{e 4.54}$$

It should be noted and will be used later that, for any $0 \leq x \leq 1$,

$$x \leq f_\delta(x) + g_{\delta_1}(x) \ \text{ for all } \ 0 < \delta < \delta_1/8. \tag{e 4.55}$$

Note, for all $k > n+1 \geq n > n_0$, that $c_n \perp (1-E_k)^{1/2}P_1(1-E_k)^{1/2}$,

$$a_k \lesssim c_n \ \text{ and } \ (1-E_k)^{1/2}P_2(1-E_k)^{1/2} \lesssim (1-E_k)^{1/2}P_1(1-E_k)^{1/2}. \tag{e 4.56}$$

Put $y_{k,n}' := c_n + (1-E_k)^{1/2}P_1(1-E_k)^{1/2}$, $k = 1, 2, ...$

There exists a function $\chi \in C_0((0, \|a\|)$ with $0 \leq \chi \leq 1$ such that $\chi(f_{\eta/2}) = f_\eta$. For any $\varepsilon > 0$, there exists $\delta_2 > 0$ such that, if $0 \leq e_1, e_2 \leq 1$ be elements in a C^*-algebra with $\|e_1 - e_2\| < \delta_2$, then $\|\chi(e_1) - \chi(e_2)\| < \varepsilon/32$. By Lemma 4.6, for any fixed $k \geq n+1 > n \geq n_0$, there are $\delta_k > 0$ and a unitary $V \in M_{2r}(\tilde{B})$ and $h_k \in \mathrm{Her}(f_{\delta_k}(y_{k,n}'))_+$ with $\|h_k\| \leq 1$ such that

$$\|V^* f_{\eta/2}(x_k)V - h_k\| < \min\{\varepsilon/32, \delta_2\}. \tag{e 4.57}$$

By (e 4.43), choose $k_{m,1} \geq k_0$ such that, for all $k \geq k_{m,1}$,

$$\|\chi(f_{\eta/2}(x_k)) - \chi(f_{\eta/2}(a))\| < \varepsilon/32. \tag{e 4.58}$$

Thus we have

$$\|V^* f_\eta(a)V - \chi(h_k)\| < 3\varepsilon/32. \tag{e 4.59}$$

Recall (see (e 4.39)), for $n > n_0$ and $k \geq \max\{k_{m,1}, k_0, n+1\}$,

$$y_k' = E_n^{1/2}b_1E_n^{1/2} + (1-E_k)^{1/2}P_1(1-E_k)^{1/2} = E_n^{1/2}b_1E_n^{1/2} + P_1(1-E_k)P_1 \tag{e 4.60}$$
$$\leq E_n^{1/2}b_1E_n^{1/2} + P_1(1-E_n)P_1 = E_n^{1/2}b_1E_n^{1/2} + (1-E_n)^{1/2}P_1(1-E_n)^{1/2} = y_n. \tag{e 4.61}$$

By (e 4.55),

$$y_n \leq f_{\delta_1/8}(y_n) + g_{\delta_1}(y_n) := \bar{y}_n \tag{e 4.62}$$

Thus $h_k \in \mathrm{Her}(\bar{y}_n)$. Choose $\varepsilon' > 0$ such that

$$\|f_{\varepsilon'}(\bar{y}_n)\chi(h_k)f_{\varepsilon'}(\bar{y}_n) - \chi(h_k)\| < \varepsilon/16. \tag{e 4.63}$$

By (e 4.53) and the strict comparison of B, $g_{\delta_1}(y_n) \lesssim b_0$. Recall $b_1 \perp b_0$ and $f_{\delta_1/8}(y_n) \lesssim b_1$. By applying Lemma 4.6 again, we obtain a unitary $W \in M_{2r}(\tilde{B})$ and $\bar{h} \in \mathrm{Her}(b_1 + b_0) \subset \mathrm{Her}(b)_+$ such that

$$\|W^* f_{\varepsilon'}(\bar{y}_n)W - \bar{h}\| < \varepsilon/8. \tag{e 4.64}$$

Let $U = VW$. Then, by (e 4.59), (e 4.63),

$$U^* f_\eta(a)U \approx_{3\varepsilon/32} W^*\chi(h_k)W \approx_{\varepsilon/16} W^* f_{\varepsilon'}(\bar{y}_n)\chi(h_k)f_{\varepsilon'}(\bar{y}_n)W \tag{e 4.65}$$
$$= W^* f_{\varepsilon'}(\bar{y}_n)WW^*\chi(h_k)WW^* f_{\varepsilon'}(\bar{y}_n)W \approx_{\varepsilon/4} \bar{h}(W^*\chi(h_k)W)\bar{h}. \tag{e 4.66}$$

Note that $\bar{h}(W^*\chi(h_k)W)\bar{h} \in \mathrm{Her}(b)$. This proves the first part of the lemma. To see the last part, let $0 < \sigma < 1/2$, the first part and Proposition of 2.2 of [33] imply that, for large n,

$$f_\sigma(U_n^* f_\eta(a)U_n) \lesssim b. \tag{e 4.67}$$

It follows that $f_\sigma(f_\eta(a)) \sim U_n^* f_\sigma(f_\eta(a))U_n = f_\sigma(U_n^* f_\eta(a)U_n) \lesssim b$ for all $0 < \sigma < 1/2$. Hence $f_\eta(a) \lesssim b$ (for all $0 \le \eta < 1/2$) which implies $a \lesssim b$. □

Lemma 4.10. *Let B be as in 4.2 and let $a \in M_r(\tilde{B})_+$ with $0 \le a \le 1$. Then there exists a sequence $0 \le a_n \le 1$ in $\mathrm{Her}(a)$ such that $[a_n] \le [a_{n+1}]$, $a = \sup\{[a_n] : n \in \mathbb{N}\}$ and $\lim_{n\to\infty} \omega([a_n]) = 0$.*

Proof. If there is a sequence $t_n \in (0, 1)$ such that $t_{n+1} < t_n$ and $\lim_{n\to\infty} t_n = 0$ and $t_n \notin \mathrm{sp}(a)$ for all n, then one obtains an increasing sequence of projections $\{p_n\}$ such that $p_n \in \mathrm{Her}(a)$, and such that for any $0 < \varepsilon < 1$, $f_\varepsilon(a) \le p_n$ for all sufficiently large n. Let $a_n := p_n$. Then $\omega([p_n]) = 0$ and $[a] = \sup\{[a_n] : n \in \mathbb{N}\}$. Thus we assume that $[0, \eta_0] \subset \mathrm{sp}(a)$ for some $\eta_0 \in (0, 1]$.

As in the proof of Lemma 4.9, we may assume that, for some integer $m \ge 1$,

$$\pi^B_{\mathbb{C}}(a) = \mathrm{diag}(\overbrace{1, 1, ..., 1}^{m}, 0, \cdots, 0) := p. \tag{e 4.68}$$

Let $P := \mathrm{diag}(\overbrace{1_{\tilde{B}}, 1_{\tilde{B}}, ..., 1_{\tilde{B}}}^{m}, 0, ..., 0)$. Let $\{e_n\}$ and $\{E_n\}$ be as in the proof 4.9 of (see (e 4.38)). Note that $E_kP = PE_k$ for all k. As in the proof of 4.9, if $0 < \eta < \eta_0/16$, then

$$\lim_{k\to\infty} \|(E_k^{1/2} a E_k^{1/2} + (1 - E_k)^{1/2} P(1 - E_k)^{1/2} - a\| = 0 \text{ and} \tag{e 4.69}$$
$$\lim_{k\to\infty} \|(E_k^{1/2} f_\eta(a) E_k^{1/2} + (1 - E_k)^{1/2} P(1 - E_k)^{1/2} - f_\eta(a)\| = 0. \tag{e 4.70}$$

Note $\pi^B_{\mathbb{C}}((1 - E_k)^{1/2} P(1 - E_k)^{1/2}) = p = \pi^B_{\mathbb{C}}(f_\eta(a))$. Note also that, since $f_\eta(a)^{1/2} E_k f_\eta(a)^{1/2} \nearrow f_\eta(a)$ (in the strict topology), $(E_k^{1/2} f_\eta(a) E_k^{1/2})\widehat{\ } \nearrow \widehat{f_\eta(a)}$ uniformly on $T(B)$ (by Dini's theorem). We may therefore assume that, if $k \ge k_\eta$ (for some $k_\eta \ge 1$),

$$(E_k^{1/2} f_\eta(a) E_k^{1/2})\widehat{\ }(\tau) > \widehat{f_\eta(a)}(\tau) - \sigma(\eta)/16 \text{ for all } \tau \in T(B), \tag{e 4.71}$$

where $\sigma(\eta) = \min\{\inf\{\tau(f_\eta(a)) - d_\tau(f_{4\eta}(a)) : \tau \in T(B)\}, \eta/16\} > 0$ (recall $[0, \eta_0] \subset \mathrm{sp}(a)$). Moreover, since $M_r(B)$ has continuous scale, we may assume, for all $k \ge k_\eta$,

$$[1 - E_{k+1}]\widehat{\ } \le (1 - E_k)\widehat{\ } < \sigma(\eta)/16. \tag{e 4.72}$$

Put $a_{\eta,k} := E_k^{1/2} f_\eta(a) E_k^{1/2}$. Choose $0 < \delta(\eta) < \sigma(\eta)/16r$. Then, by (e 4.71), for any $k \geq k_{\eta,1}$ for some $k_{\eta,1} \geq k_\eta + 1$,

$$\tau(f_{\delta(\eta)}(a_{\eta,k})) \geq \tau((a_{\eta,k} - \delta(\eta))_+) > \tau(a_{\eta,k} - \delta(\eta)) \\ = \tau(a_{\eta,k}) - r\delta(\eta) > \tau(f_\eta(a)) - \sigma(\eta)/8 \text{ for all } \tau \in T(B). \tag{e 4.73}$$

By (e 4.70) and Proposition 2.2 of [33], there is $k_{\eta,2} \geq k_{\eta,1}$ such that, for all $k \geq k_{\eta,2}$, there is $x_{\delta/8,\eta} \in \mathrm{Her}(f_\eta(a))$ such that

$$f_{\delta(\eta)/8}((1-E_k)^{1/2}P(1-E_k)^{1/2} + E_k^{1/2} f_\eta(a) E_k^{1/2}) \sim x_{\delta/8,\eta}. \tag{e 4.74}$$

Since B is stably projectionless, for any nonzero $0 \leq b \leq 1$ in $M_r(B)$, $\mathrm{sp}(b) = [0,1]$. Thus

$$d_\tau(f_{\delta(\eta)}(a_{\eta,k}))) < \tau(f(a_{\eta,k})) < d_\tau(f_{\delta(\eta)/2}(a_{\eta,k})) \text{ for all } \tau \in T(B), \tag{e 4.75}$$

where $0 \leq f \leq 1$ is in $C_0((0,1])$ such that $f(t) = 1$ for $t \in [\delta(\eta)/2, 1]$, $f(t) = 0$ for $t \in (0, \delta(\eta)/4]$. Since $\mathrm{Cu}(B) = \mathrm{LAff}_+(T(B))$, there is $c_{k,\eta,\delta(\eta)} \in M_r(B)_+$ with $\|c_{k,\eta,\delta(\eta)}\| \leq 1$ such that, for all $\tau \in T(B)$, $d_\tau(c_{n,\eta,\delta(\eta)}) = \tau(f(a_{\eta,k}))$ which is continuous on $T(B)$. Since B has strict comparison, by (e 4.75), $c_{k,\eta,\delta(\eta)} \lesssim f_{\delta(\eta)/2}(a_{\eta,k})$. Since $M_r(B)$ has almost stable rank one, by Lemma 2.12, we may assume that $c_{k,\eta,\delta(\eta)} \in \mathrm{Her}(f_{\delta(\eta)/2}(a_{\eta,k}))$.

Note that $(1-E_{k+1})E_k = 0$ and $P(1-E_k)^{1/2} = (1-E_k)^{1/2}P$ for all k. In particular, $P(1-E_{k+1})P = (1-E_{k+1})^{1/2}P(1-E_{k+1})^{1/2} \perp a_{\eta,k}$. By Lemma 2.13, there exists $z \in M_r(\tilde{B})$ such that (see also (e 4.74))

$$f_{\delta(\eta)/2}((1-E_{k+1})^{1/2}P(1-E_{k+1})^{1/2}) + c_{k,\eta,\delta(\eta)} \leq f_{\delta(\eta)/4}(P(1-E_{k+1})P) + f_{\delta(\eta)/4}(a_{\eta,k}) \\ = f_{\delta(\eta)/4}(P(1-E_{k+1})P + a_{\eta,k}) \sim z^*z, \text{ and} \tag{e 4.76}$$

$$z^*z \lesssim f_{\delta(\eta)/8}(P(1-E_k)P + a_{\eta,k}) \tag{e 4.77}$$

$$= f_{\delta(\eta)/8}((1-E_k)^{1/2}P(1-E_k)^{1/2} + E_k^{1/2} f_\eta(a) E_k^{1/2}) \sim x_{\delta/8,\eta} \in \mathrm{Her}(f_\eta(a)). \tag{e 4.78}$$

Define $b_{k,\eta,\delta(\eta)} := f_{\delta(\eta)/2}((1-E_{k+1})^{1/2}P(1-E_{k+1})^{1/2}) + c_{k,\eta,\delta(\eta)}$ for $k \geq k_{\eta,2}$. From the displays above, there is $y_{k,\eta,\delta(\eta)} \in \mathrm{Her}(f_\eta(a))$ such that $b_{k,\eta,\delta} \sim y_{k,\eta,\delta(\eta)}$. By (e 4.75) and (e 4.73), we have, for $k \geq k_{\eta,2}$, and for all $\tau \in T(B)$,

$$d_\tau(y_{k,\eta,\delta(\eta)}) = d_\tau(b_{k,\eta,\delta(\eta)}) > d_\tau(c_{k,\eta,\delta(\eta)}) \tag{e 4.79}$$

$$\geq \widehat{[f_{\delta(\eta)}(E_k^{1/2} f_\eta(a) E_k^{1/2})]}(\tau) \geq \tau(f_{\delta(\eta)}(a_{\eta,k})) > \tau(f_\eta(a)) - \sigma(\eta)/8. \tag{e 4.80}$$

Since $\widehat{[c_{k,\eta,\delta(\eta)}]}$ is continuous on $T(B)$, for $k \geq k_{\eta,\delta}$, by (e 4.72)
(recall $f_{\delta(\eta)/2}((1-E_{k+1})^{1/2}P(1-E_{k+1})^{1/2}) \perp c_{k,\eta,\delta(\eta)}$),

$$\omega([y_{k,\eta,\delta(\eta)}]) < \sigma(\eta)/16 \leq \eta/32. \tag{e 4.81}$$

Combing (recall the definition of $\sigma(\eta)$) (e 4.81) and (e 4.80), for all $\tau \in T(B)$,

$$d_\tau(f_{8\eta}(a)) + 4\omega([y_{k,\eta,\delta(\eta)}]) \leq \tau(f_{4\eta}(a)) + 4\omega([y_{k,\eta,\delta(\eta)}]) < \tau(f_\eta(a)) - \sigma(\eta) + 4\omega([y_{k,\eta,\delta(\eta)}]) \tag{e 4.82}$$

$$< \tau(f_\eta(a)) - 5\sigma(\eta)/16 < d_\tau(y_{k,\eta,\delta(\eta)}). \tag{e 4.83}$$

We also have $[\pi^B_{\mathbb{C}}(f_{2\eta}(a))] \leq [\pi^B_{\mathbb{C}}(f_\eta(a))] = p = [\pi^B_{\mathbb{C}}((1-E_{k_{\eta,2}})P)] = [\pi^B_{\mathbb{C}}(y_{k_{\eta,2},\eta,\delta(\eta)})]$. By Lemma 4.9,

$$f_{8\eta}(a) \lesssim y_{k_{\eta,2},\eta,\delta(\eta)}. \tag{e 4.84}$$

For each fixed $0 < \eta < 1/8$, there exists $\mu_\eta > 0$ such that (recall $[0, \eta_0] \subset \mathrm{sp}(a)$)

$$\tau(f_{\eta/4}(a)) > d_\tau(f_\eta(a)) + \mu_\eta \text{ for all } \tau \in T(B). \tag{e 4.85}$$

Choose $0 < \eta' < \eta/16$ such that $\eta' < \mu_\eta/4$. Then, for $k \geq k_{\eta',2}$, and for all $\tau \in T(B)$, by (e 4.80), (e 4.85), (and recall the definition of $\sigma(\eta')$ and $y_{k_{\eta,2},\eta,\delta(\eta)} \in \mathrm{Her}(f_\eta(a))$), and (e 4.81),

$$[y_{k_{\eta',2},\eta',\delta(\eta')}]\widehat{\ }(\tau) \geq \tau(f_{\eta'}(a)) - \sigma(\eta')/8 \geq d_\tau(f_\eta(a)) + \mu_\eta/2 \geq [y_{k_{\eta,2},\eta,\delta(\eta)}]\widehat{\ }(\tau) + \mu_\eta/2 \tag{e 4.86}$$

$$> [y_{k_{\eta,2},\eta,\delta(\eta)}]\widehat{\ }(\tau) + \eta' \geq [y_{k_{\eta,2},\eta,\delta(\eta)}]\widehat{\ }(\tau) + 4\omega([y_{k_{\eta',2},\eta',\delta(\eta')}]) \text{ for all } \tau \in T(B). \tag{e 4.87}$$

Recall $\pi_{\mathbb{C}}^B(f_{\delta(\eta)}((1-E_k)^{1/2}P(1-E_k)^{1/2})) = p$ for all k and for all $\delta(\eta) < 1/2$. It follows from Lemma 4.9 (or from Lemma A.3 of [17])

$$f_{8\eta}(a) \lesssim y_{k_{\eta,2},\eta,\delta(\eta)} \lesssim y_{k_{\eta',2},\eta',\delta(\eta')}. \tag{e 4.88}$$

Thus, we obtain a sequence $\{c_n\}$ which is a subsequence of $\{y_{k_{\eta,2},\eta,\delta(\eta)}\} \in \mathrm{Her}(a)$ (with $\eta \to 0$) such that

$$[c_n] \leq [c_{n+1}] \text{ and } \lim_{n\to\infty} \omega([c_n]) = 0 \,(\text{see (e 4.81)}). \tag{e 4.89}$$

Put $x = \sup\{[c_n] : n \in \mathbb{N}\}$ (see Theorem 1 of [11]). Then, by (e 4.84), $[f_{8\eta}(a)] \leq x$ for all $0 < \eta < \min\{\eta_0, 1/16\}$. It follows that $[a] \leq x$. Since each $c_n \in \mathrm{Her}(a)$, $x \leq [a]$. It follows that $x = [a]$. □

In the following statement, it should be noted that we do not assume that $\tilde{B}$ has almost stable rank one. One of the features of the following statement is the existence of the unitaries U_n which compensates the absence of the cancellation for our late purposes.

Theorem 4.11. *Let B be as in 4.2 and $a, b \in M_r(\tilde{B})_+$ (where $r \geq 1$ is an integer). Suppose that $\pi_{\mathbb{C}}^B(a) \lesssim \pi_{\mathbb{C}}^B(b)$, and*

$$d_\tau(a) < d_\tau(b) \text{ for all } \tau \in T(B). \tag{e 4.90}$$

Then, for any $1 > \eta > 0$, there exists a sequence of unitaries $U_n \in M_{2r}(\tilde{B})$ and a sequence of elements $h_n \in \mathrm{Her}(b)_+$ such that

$$\lim_{n\to\infty} \|U_n^* f_\eta(a) U_n - h_n\| = 0. \tag{e 4.91}$$

Proof. First consider the case that $[a] = [p]$ for some projection $p \in M_r(\tilde{B})$. Then $d_\tau(a) = \tau(p)$ is continuous on $T(B)$. Put

$$\sigma := (1/2)\inf\{d_\tau(b) - \tau(p) : \tau \in T(B)\} > 0. \tag{e 4.92}$$

Since $\tau(f_{1/2^n}(b)) \nearrow d_\tau(b)$, as $n \to \infty$, there exists $n_0 \geq 1$ such that, for all $n \geq n_0$,

$$d_\tau(a) + \sigma < \tau(f_{1/2^n}(b)) \text{ for all } \tau \in T(B) \text{ and } [\pi_{\mathbb{C}}^B(f_{1/2^n}(b))] = [\pi_{\mathbb{C}}^B(b)]. \tag{e 4.93}$$

By Lemma 4.10, there exists a sequence of elements $b_n \in \mathrm{Her}(b)_+$ with $0 \leq b_n \leq 1$ and an integer $N \geq 1$ such that, for all $n \geq N$ (as $[f_{1/2^{n_0+1}}(b)] \ll [b]$),

$$f_{1/2^{n_0+1}}(b) \lesssim b_n \text{ and } \lim_{n\to\infty} \omega([b_n]) = 0. \tag{e 4.94}$$

Thus, there exists $n_1 \geq N + n_0$, for all $n \geq n_1$

$$d_\tau(a) + 4\omega([b_n]) < d_\tau(b_n) \text{ for all } \tau \in T(B) \text{ and } [\pi_{\mathbb{C}}^B(a)] \leq [\pi_{\mathbb{C}}^B(b_n)]. \tag{e 4.95}$$

Applying Lemma 4.9, for any $\eta > 0$, there exist a sequence of unitaries $U_n \in M_{2r}(\tilde{B})$ and a sequence of elements $h_n \in \mathrm{Her}(b_{n_1})_+$ such that

$$\lim_{n\to\infty} \|U_n^* f_\eta(a) U_n - h_n\| = 0. \tag{e 4.96}$$

Note that $h_n \in \mathrm{Her}(b_{n_1})_+ \subset \mathrm{Her}(b)_+$.

Next consider the case that $[a]$ cannot be represented by a projection. It follows that 0 is not an isolated point.

Fix $0 < \eta < 1$. Choose $0 < \varepsilon < \eta/4$, by Lemma 4.8, there exists $\sigma_0 > 0$ such that

$$d_\tau(f_{\varepsilon/2}(a)) + \sigma_0 < d_\tau(f_{\eta/4}(a)) < d_\tau(b) \text{ for all } \tau \in T(B). \tag{e 4.97}$$

Choose $b_n \in \mathrm{Her}(b)_+$ above. Then, there exists $n_2 \geq 1$ such that, for all $n \geq n_2$,

$$d_\tau(f_{\varepsilon/2}(a)) + 4\omega([b_n]) < d_\tau(b_n) \text{ and } [\pi_{\mathbb{C}}^B(f_{\varepsilon/2}(a))] \leq [\pi_{\mathbb{C}}^B(b_n)]. \tag{e 4.98}$$

Applying Lemma 4.9, one obtains a sequence of unitaries $U_n \in M_{2r}(\tilde{B})$ and a sequence of elements $h_n \in \mathrm{Her}(b_{n_2})_+ \subset \mathrm{Her}(b)_+$ such that

$$\lim_{n\to\infty} \|U_n^* f_\eta(a) U_n - h_n\| = 0. \tag{e 4.99}$$

Theorem follows. □

We now arrive at the following theorem (see Theorem A.6 of [17]).

Theorem 4.12. *Let B be as in 4.2. Then, for any $a, b \in (\tilde{B} \otimes \mathcal{K})_+$, if $[\pi_{\mathbb{C}}^B(a)] \leq [\pi_{\mathbb{C}}^B(b)]$ and*

$$d_\tau(a) < d_\tau(b) \text{ for all } \tau \in T(B), \tag{e 4.100}$$

then $a \lesssim b$. Moreover, if $[a]$ is not represented by a projection, then $d_\tau(a) \leq d_\tau(b)$ for all $\tau \in T(\tilde{B})$ implies that $a \lesssim b$.

Proof. For the first part, we note that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_\tau(f_\varepsilon(a)) < d_\tau(f_\delta(b)) \text{ for all } \tau \in T(B) \text{ and } [\pi_{\mathbb{C}}^B(f_\varepsilon(a))] \leq [\pi_{\mathbb{C}}^B(f_\delta(b))]. \tag{e 4.101}$$

With this observation, we reduce the general case to the case that $a, b \in M_r(\tilde{B})_+$ with $0 \leq a,\, b \leq 1$.

For this case, for any $0 < \eta < 1/2$, by Lemma 4.11, there is $h \in \mathrm{Her}(b)_+$ and a unitary $U \in M_{2r}(\tilde{B})$ such that

$$\|U^* f_{\eta/4}(a) U - h\| < \eta/8. \tag{e 4.102}$$

By Proposition 2.2 of [33], this implies that

$$f_{\eta/4}(f_{\eta/4}(a)) \sim U^* f_{\eta/4}((f_{\eta/4}(a))U = f_{\eta/4}(U^* f_{\eta/4}(a)U) \lesssim h \lesssim b. \tag{e 4.103}$$

Since this holds for all $0 < \eta < 1/2$, one has $a \lesssim b$.

Now suppose that

$$d_\tau(a) \leq d_\tau(b) \text{ for all } \tau \in T(\tilde{B}). \tag{e 4.104}$$

If $[a]$ is not represented by a projection, then, by Lemma 4.8, for any $1 > \varepsilon > 0$,

$$d_\tau(f_\varepsilon(a)) < d_\tau(b) \text{ for all } \tau \in T(B) \text{ and } [\pi_{\mathbb{C}}^B(f_\varepsilon(a))] \leq [\pi_{\mathbb{C}}^B(b)]. \tag{e 4.105}$$

By what has been proved above, $f_\varepsilon(a) \lesssim b$ for all $1 > \varepsilon > 0$. Therefore $a \lesssim b$.

□

Combining Theorem 4.12 and Lemma 4.4, we have the following description of the $\mathrm{Cu}(\tilde{B})$. Note that all finite exact separable simple stably projectionless $\mathcal{Z}$-stable C^*-algebras satisfy the assumption of the corollary below.

Corollary 4.13. *Let B be a separable stably projectionless simple C^*-algebra with continuous scale such that $M_n(A)$ has almost stable rank one (for all $n \in \mathbb{N}$), and such that $QT(B) = T(B)$ and $\mathrm{Cu}(B) = \mathrm{LAff}_+(T(B))$. Then, $\mathrm{Cu}(\tilde{B}) = (V(\tilde{B}) \setminus \{0\}) \sqcup \mathrm{LAff}_+(T(\tilde{B}))^{\diamond}$.*

Remark 4.14. If both x and y are not compact in $\mathrm{Cu}(\tilde{B})$ and $x \mathrel{\mathring{=}} y$, or equivalently $x + k[1_{\tilde{B}}] = y + k[1_{\tilde{B}}]$ in $\mathrm{Cu}(\tilde{B})$ for some integer k, then, by Theorem 4.12, $x = y$. So $\mathrm{Cu}(\tilde{B})$ has the weak version of cancellation. However, we still do not have the cancellation for projections. In other words, if $p \oplus e \sim q \oplus e$ for some nonzero projection e, we do not know that $p \sim q$. Nevertheless, if $[p \oplus e] + x \le [q \oplus e]$ for some $x \in \mathrm{Cu}(\tilde{A})_+ \setminus \{0\}$, then $[p] \le [q]$, by Theorem 4.12.

5 Approximation

In this section we will present Lemma 5.3 (see also the last part of Remark 5.4).

Definition 5.1. Let A and B be C^*-algebras and $\lambda : \mathrm{Cu}^{\sim}(A) \to \mathrm{Cu}^{\sim}(B)$ be a morphism in **Cu** (see [30]). Suppose that $\varphi_n : A \to B$ is a sequence of homomorphisms. We say $\mathrm{Cu}(\varphi_n)$ converges to λ and write $\lim_{n\to\infty} \mathrm{Cu}(\varphi_n) = \lambda$, if, for any finite subset $G \subset \mathrm{Cu}^{\sim}(A)$, there exists $N \ge 1$ such that, for all $n \ge N$,

$$\mathrm{Cu}^{\sim}(\varphi_n)(x) \le \lambda(y) \ \text{ and } \ \lambda(x) \le \mathrm{Cu}^{\sim}(\varphi_n)(y), \tag{e 5.1}$$

whenever $x, y \in G$ and $x \ll y$.

Let $G_0 \subset K_0(A) \subset \mathrm{Cu}^{\sim}(A)$ (see 6.1 of [32]) be a finite subset. Then $\lim_{n\to\infty} \mathrm{Cu}^{\sim}(\varphi_n) = \lambda$ implies that, there is an integer $n_0 \ge 1$ such that, for all $n \ge n_0$,

$$\mathrm{Cu}^{\sim}(\varphi_n)(x) = \lambda(x) \ \text{ for all } \ x \in G_0 \tag{e 5.2}$$

as $x \ll x$ in $\mathrm{Cu}^{\sim}(A)$.

We write $\lim^{w}_{n\to\infty} \mathrm{Cu}^{\sim}(\varphi_n) = \lambda$, if for any finite subset $G \subset \mathrm{Cu}^{\sim}(A)$, there exists $n_0 \ge 1$ such that, for all $n \ge n_0$,

$$\mathrm{Cu}^{\sim}(\varphi_n)(z) = \lambda(z) \ \text{ for all } \ z \in G \cap K_0(A) \ \text{ and} \tag{e 5.3}$$

$$\mathrm{Cu}^{\sim}(\varphi_n)(x) \le \lambda(y) \ \text{ and } \ \lambda(x) \le \mathrm{Cu}^{\sim}(\varphi_n)(y), \tag{e 5.4}$$

whenever $x, y \in G$ and $x \ll y$ and both x and y are not compact.

Lemma 5.2. *Let C be a separable C^*-algebra of stable rank one and B be a C^*-algebra with finite stable rank. Suppose that $\lambda : \mathrm{Cu}^{\sim}(C) \to \mathrm{Cu}^{\sim}(B)$ is a morphism in **Cu** and there exists a sequence of homomorphisms $\varphi_n : C \to B$ such that*

$$\lim_{n\to\infty} \mathrm{Cu}^{\sim}(\varphi_n) = \lambda. \tag{e 5.5}$$

Suppose that $\psi_n : C \to B$ is a sequence of homomorphisms such that

$$\lim_{n\to\infty} \|\psi_n(a) - \varphi_n(a)\| = 0 \ \text{ for all } \ a \in C. \tag{e 5.6}$$

Then

$$\lim_{n\to\infty} \mathrm{Cu}^{\sim}(\psi_n) = \lambda. \tag{e 5.7}$$

Proof. Let $G \subset \mathrm{Cu}^\sim(C)$ be a finite subset. Let $S = \{(f,g) : f, g \in G, f \ll g\}$.

Suppose that $(f,g) \in S$. We claim, in this case, that there is $h \in \mathrm{Cu}^\sim(C)$ such that

$$f \ll h \ll g. \tag{e 5.8}$$

Recall that C has stable rank one. We may assume that $f = [a^f] - m_f[1_{\tilde{C}}]$ and $g = [a^g] - m_g[1_{\tilde{C}}]$, where $a^f, a^g \in M_r((\tilde{C}))_+$ with $\|a^f\| \le 1$ and $\|a^g\| \le 1$ for some integer $r \ge 1$, and, rank of $\pi_{\mathbb{C}}^C(a^f)$ is $m_f \le r$, and rank of $\pi_{\mathbb{C}}^C(a^g)$ is $m_g \le r$. Therefore

$$[a^f \oplus 1_{m_g}] \ll [a^g \oplus 1_{m_f}] \tag{e 5.9}$$

(in the $\mathrm{Cu}(\tilde{C})$), where 1_{m_f} and 1_{m_g} are identities of $M_{m_f}(\tilde{C})$ and $M_{m_g}(\tilde{C})$ respectively.

By (e 5.9), there is $1/2 > \varepsilon > 0$ such that

$$[a^f \oplus 1_{m_g}] \ll [f_\varepsilon(a^g) \oplus 1_{m_f}] \text{ and } [f_\varepsilon(a^g)] \ll [a^g]. \tag{e 5.10}$$

Moreover, by choosing smaller ε, we may assume that $\pi_{\mathbb{C}}^C(f_\varepsilon(a^g)) = f_\varepsilon(\pi_{\mathbb{C}}^C(a^g))$ has the same rank as that of $[\pi_{\mathbb{C}}^C(a^g)] = m_g$. Put $a^h = f_\varepsilon(a^g)$ and $h = [a^h] - m_g[1_{\tilde{C}}]$. Then

$$f \ll h \ll g. \tag{e 5.11}$$

Define $\bar{f} = \mathrm{diag}(a^f, 1_{m_g})$, $\bar{h} = \mathrm{diag}(a^h, 1_{m_f})$ and $\bar{g} = \mathrm{diag}(a^g, 1_{m_f})$. Note that, in $\mathrm{Cu}(\tilde{C})$,

$$\bar{f} \ll \bar{h} \ll \bar{g}. \tag{e 5.12}$$

Choose $0 < \delta < \varepsilon/4$ such that

$$\bar{f} \le f_\delta(\bar{h}) \le \bar{h} \text{ and } \bar{h} \le f_\delta(\bar{g}) \le \bar{g}. \tag{e 5.13}$$

Let $\varphi_n^\sim, \psi_n^\sim : M_r(\tilde{C}) \to M_r(\tilde{B})$ be the (unital) extensions of φ_n and ψ_n, respectively. We claim that, for each $(f,g) \in S$, there is an integer $N \ge 1$ such that, for all $n \ge N$,

$$\mathrm{Cu}^\sim(\psi_n)(f) \le \lambda(g) \text{ and } \lambda(f) \le \mathrm{Cu}^\sim(\psi_n). \tag{e 5.14}$$

Write $\lambda(f) = \lambda(f)_+ - m_{\lambda,f}[1_{\tilde{B}}]$ and $\lambda(g) = \lambda(g)_+ - m_{\lambda,g}[1_{\tilde{B}}]$, where $\lambda(f)_+ = [a_{\lambda,f}]$ and $\lambda(g)_+ = [a_{\lambda,g}]$ for some $a_{\lambda,f}, a_{\lambda,g} \in M_r(\tilde{B})_+$ (by enlarge r if necessary), and $[\pi_{\mathbb{C}}^B(a_{\lambda,f})] = m_{\lambda,f}$ and $[\pi_{\mathbb{C}}^B(a_{\lambda,g})] = m_{\lambda,g}$ are integers.

Note, by (e 5.6), we have

$$\lim_{n\to\infty} \|\psi_n^\sim(c) - \varphi_n^\sim(c)\| = 0 \text{ for all } c \in M_r(\tilde{C}). \tag{e 5.15}$$

Then, by (e 5.13) and by repeated application of Proposition 2.2 of [33], there exists an integer $N \ge 1$ such that, if $n \ge N$,

$$\psi_n^\sim(\bar{f}) \lesssim \psi_n^\sim(f_\delta(\bar{h})) \lesssim \varphi_n^\sim(\bar{h}) \text{ and } \varphi_n^\sim(\bar{h}) \lesssim \varphi_n^\sim(f_\delta(\bar{g})) \lesssim \psi_n^\sim(\bar{g}). \tag{e 5.16}$$

Assume that B has stable rank K. Since $\lim_{n\to\infty} \mathrm{Cu}^\sim(\varphi_n) = \lambda$, we may also assume, if $n \ge N$,

$$[\varphi_n^\sim(a^h)] + (m_{\lambda,g} + K)[1_{\tilde{B}}] \le \lambda(g)_+ + (m_g + K)[1_{\tilde{B}}] \text{ and} \tag{e 5.17}$$

$$\lambda(f)_+ + (m_g + K)[1_{\tilde{B}}] \le [\varphi_n^\sim(a^h)] + (m_{\lambda,f} + K)[1_{\tilde{B}}] \tag{e 5.18}$$

for all $(f,g) \in S$. Combining (e 5.16), (e 5.17) and (e 5.18), we obtain

$$\begin{aligned}
&[\psi_n^\sim(a^f)] + (m_g + m_{\lambda,g} + K)[1_{\tilde{B}}] = [\psi_n^\sim(\bar{f})] + (m_{\lambda,g} + K)[1_{\tilde{B}}] \le [\psi_n^\sim(f_\delta(\bar{h})] + (m_{\lambda,g} + K)[1_{\tilde{B}}] \\
&\quad \le [\varphi_n^\sim(a^h)] + (m_f + m_{\lambda,g} + K)[1_{\tilde{B}}] \le \lambda(g)_+ + (m_g + K)[1_{\tilde{B}}] + m_f[1_{\tilde{B}}] \text{ and} \\
&\lambda(f)_+ + (m_f + m_g + K)[1_{\tilde{B}}] \le [\varphi_n^\sim(a^h)] + (m_{\lambda,f} + m_f + K)[1_{\tilde{B}}] = [\varphi_n^\sim(\bar{h})] + (m_{\lambda,f} + K)[1_{\tilde{B}}] \\
&\quad \le [\psi_n^\sim(\bar{g})] + (m_{\lambda,f} + K)[1_{\tilde{B}}] = [\psi_n^\sim(a^g)] + (m_f + m_{\lambda,f} + K)[1_{\tilde{B}}].
\end{aligned}$$

Thus, for all $n \geq N$, and, for all $(f, g) \in S$,

$$\mathrm{Cu}^{\sim}(\psi_n)(f) \leq \lambda(g) \text{ and } \lambda(f) \leq \mathrm{Cu}^{\sim}(\psi_n)(g).$$

$\square$

Lemma 5.3. *Let C be a separable semiprojective C^*-algebra with a strictly positive element e_C and B be as in 4.2.*

*(1) Let $\lambda : \mathrm{Cu}^{\sim}(C) \to \mathrm{Cu}^{\sim}(\tilde{B})$ be a morphism in **Cu** with $\lambda([e_C]) \leq [b]$ for some $b \in M_N(\tilde{B})_+$ (and $N \geq 1$), and let $\varphi_k : C \to M_N(\tilde{B})$ be a sequence of homomorphisms such that $\lim_{k\to\infty} \mathrm{Cu}^{\sim}(\varphi_k) = \lambda$, then exists a sequence of homomorphisms $\psi_k : C \to \overline{bM_N(\tilde{B})b}$ such that*

$$\lim_{k\to\infty} \mathrm{Cu}^{\sim}(\psi_k) = \lambda, \tag{e 5.19}$$

if, in addition,

(i) $\widehat{\lambda([e_C])}(\tau) < \widehat{[b]}(\tau)$ for all $\tau \in T(B)$, or

(ii) $\lambda([e_C])$ is not a compact element in $\mathrm{Cu}^{\sim}(\tilde{B})$.

*(2) If $\lambda : \mathrm{Cu}^{\sim}(C) \to \mathrm{Cu}^{\sim}(B)$ is a morphism in **Cu**, $\lambda([e_C]) \leq [b]$ for some $b \in M_N(B)_+$, and there exists a sequence of homomorphisms $\varphi_k : C \to M_N(B)$ such that $\lim_{k\to\infty} \mathrm{Cu}^{\sim}(\varphi_k) = \lambda$, then there exists a sequence of homomorphisms $\psi_k : C \to \overline{bM_N(B)b}$ such that*

$$\lim_{n\to\infty} \mathrm{Cu}^{\sim}(\psi_n) = \lambda. \tag{e 5.20}$$

Proof. Let us consider case (1) first. For any $\varepsilon > 0$, there exists $k(\varepsilon) \geq 1$ such that $[\varphi_k(f_{\varepsilon/4}(e_C))] \leq \lambda([f_{\varepsilon/16}(e_C)]) \leq \lambda([e_C])$ for all $k \geq k(\varepsilon)$. Put $a(k, \varepsilon) := \varphi_k(f_{\varepsilon/4}(e_C))$. For case (i), we have

$$d_\tau(a(k, \varepsilon)) < d_\tau(b) \text{ for all } \tau \in T(B). \tag{e 5.21}$$

For case (ii), let $e \in (\tilde{B} \otimes \mathcal{K})_+$ be such that $[e] = \lambda([e_C])$. Then $e \not\sim p$ for any projection. In other words, we may assume that 0 is not an isolated point in $\mathrm{sp}(e)$. Moreover, since λ is a morphism in **Cu**, it maps compact elements to compact elements. Hence $[e_C]$ cannot be represented by a projection. It follows that 0 is not an isolated point in $\mathrm{sp}(e_C)$. Choose $\eta > 0$ such that

$$[\lambda(f_{\varepsilon/16}(e_C))] \leq [f_\eta(e)]. \tag{e 5.22}$$

For any $\eta > 0$, there is a nonzero element $c \in \mathrm{Her}(e)_+$ such that $c \perp f_\eta(e)$ (see Lemma 4.8). Since B is simple, $\tau(c) > 0$ for all $\tau \in T(B)$. It follows that

$$d_\tau(f_\eta(e)) < d_\tau(b) \text{ for all } \tau \in T(B). \tag{e 5.23}$$

Thus we also have

$$d_\tau(a(k, \varepsilon)) < d_\tau(b) \text{ for all } \tau \in T(B). \tag{e 5.24}$$

Recall that $\lambda([e_C]) \leq [b]$ implies that $\mathrm{Cu}^{\sim}(\pi_{\mathbb{C}}^B) \circ \lambda([e_C]) \leq [\pi_{\mathbb{C}}^B(b)]$. Thus, in both case (i) and (ii), by Theorem 4.11, there exist a sequence of unitaries $U_n \in M_{2N}(\tilde{B})$ and a sequence of elements $h_n \in \mathrm{Her}(b)_+$ with $\|h_n\| \leq 1$ such that

$$\|U_n^* f_\varepsilon(\varphi_{k(\varepsilon)}(e_C))U_n - h_n\| < 1/2^{n+1}, \ n = 1, 2, ... \tag{e 5.25}$$

Put $\varepsilon_n > 0$ such that $\lim_{n\to\infty} \varepsilon_n = 0$. One obtains a sequence of elements $e_n \in \mathrm{Her}(b)_+$ with $\|e_n\| = 1$ and a sequence of uniaries $V_n \in M_{2N}(\tilde{B})$ such that

$$\|e_n V_n^* f_{\varepsilon_n} \varphi_{k(\varepsilon_n)}(e_C) V_n e_n - V_n^* f_{\varepsilon_n}(\varphi_{k(\varepsilon_n)}(e_C))V_n\| < 1/2^n, \ n = 1, 2, ... \tag{e 5.26}$$

Put $C_n = \overline{f_{2\varepsilon_n}(e_C)Cf_{\varepsilon_n}(e_C)}$, $\Phi_n : C \to M_{2N}(\tilde{B})$ by $\Phi_n(c) = V_n^*\varphi_{k(\varepsilon_n)}(c)V_n$, and contractive completely positive linear maps $L_n : C \to \mathrm{Her}(b)$ such that $L_n(c) = e_nV_n^*\varphi_{k(\varepsilon_n)}(f_{\varepsilon_n}(e_C)cf_{\varepsilon_n}(e_C))V_ne_n$ for $c \in C$. Then

$$\lim_{n\to\infty} \|L_n(c)L_n(c') - L_n(cc')\| = 0 \text{ for all } c, c' \in C. \tag{e 5.27}$$

Since C is semiprojective, there exists a sequence of homomorphisms $\psi_n : C \to \mathrm{Her}(b)$ such that

$$\lim_{n\to\infty} \|\psi_n(c) - L_n(c)\| = 0 \text{ for all } c \in C. \tag{e 5.28}$$

Let $\Phi_n^{\sim}, \psi_n^{\sim} : \tilde{C} \to \tilde{B}$ be the usual unitization of Φ_n and ψ_n, respectively. Then, by (e 5.26), for a fixed m, on $\mathbb{C} \cdot 1_{\tilde{C}} + C_m$,

$$\lim_{n\to\infty} \|\psi_n^{\sim}(c) - \Phi_n^{\sim}(c)\| = 0 \text{ (for all } c \in \mathbb{C} \cdot 1_{\tilde{C}} + C_m). \tag{e 5.29}$$

Note that V_n are unitaries in $M_2(\tilde{B})$. Hence $\mathrm{Cu}^{\sim}(\Phi_n) = \mathrm{Cu}^{\sim}(\varphi_{k(\varepsilon_n)})$. It follows from Lemma 5.2 that

$$\lim_{n\to\infty} \mathrm{Cu}^{\sim}(\psi_n) = \lambda. \tag{e 5.30}$$

For case (2), we work in B. By the end of 4.2, $\lambda([e_C]) \le [b]$ in $\mathrm{Cu}(B)$. Then, instead of (e 5.25), since B has almost stable rank one, by Lemma 2.10, there is, for each k, a unitary $U \in \tilde{M}_{2N}(\tilde{B})$ such that

$$U^*f_\varepsilon(\varphi_k(e_C))U \in \mathrm{Her}(b). \tag{e 5.31}$$

The rest of the proof is similar but simpler. □

Remark 5.4. Let $\varphi : C \to M_n(\tilde{B})$ be a homomorphism such that $[\varphi(e_C)] \le [b]$ in $\mathrm{Cu}^{\sim}(\tilde{B})$ for some $b \in \tilde{B}_+$, where e_C is a strictly positive element of C. Since we do not know whether $\mathrm{Cu}^{\sim}(\tilde{B})$ has the cancellation, in the case that $[\varphi(e_C)]$ is represented by a projection, there might not be any $d \in \tilde{B}_+$ such that $[d] = [\varphi(e_C)]$ in $\mathrm{Cu}^{\sim}(\tilde{B})$. In that case, there would not be any homomorphism $\psi : C \to \tilde{B}_+$ such that $\mathrm{Cu}^{\sim}(\psi) = \mathrm{Cu}^{\sim}(\varphi)$. Suppose that there is $d \in \tilde{B}_+$ such that $[d] = [\varphi(e_C)]$ in $\mathrm{Cu}^{\sim}(\tilde{B})$. We still do not know $d \sim \varphi(e_C)$ in $\mathrm{Cu}(\tilde{B})$ without knowing the cancellation in $\mathrm{Cu}^{\sim}(\tilde{B})$.

Suppose that $[\varphi(e_C)]$ is not a compact element. In an ideal situation, say there is $x \in M_n(\tilde{B})$ such that $x^*x = \varphi(e_C)$ and $xx^* \in \mathrm{Her}(d)$, then one obtains a partial isometry $v \in M_n(\tilde{B})^{**}$ such that $v^*v\varphi(c) = \varphi(c)v^*v = \varphi(c)$ for all $c \in C$ and $v\varphi(c)v^* \in \mathrm{Her}(d)$. Define $\psi : C \to \mathrm{Her}(d)$ by $\psi(c) = v\varphi(c)v^*$ for all $c \in C$. Then $\mathrm{Cu}(\psi) = \mathrm{Cu}(\varphi)$. However, $\mathrm{Cu}^{\sim}(\psi)$ may not be the same as $\mathrm{Cu}^{\sim}(\varphi)$ (see Example 6.8 below). It is crucial that we have unitaries U_n in Theorem 4.11.

Let us assume that $\lambda([e_C])$ is compact and $\lambda([e_C]) \le [b]$ for some $b \in \tilde{B}_+$. Suppose that $[\lambda([e_C]) \ne [1_{\tilde{B}}]$. Since B is stably projectionless, $\tilde{B}$ has only one nonzero projection $1_{\tilde{B}}$. To see this, let $p \in \tilde{B}$ be a nonzero projection. Then $p \notin B$. Therefore $\pi_{\mathbb{C}}^B(p) = \pi_{\mathbb{C}}^B(1_{\tilde{B}})$. This implies that $1_{\tilde{B}} - p \in B$. Since B is stably projectionless, $p = 1_{\tilde{B}}$. Therefore, in this case, $\lambda([e_C])$ cannot be represented by an element in $\tilde{B}$. Consequently, there will be no sequence of homomorphisms $\psi_k : C \to \tilde{B}$ such that $\lim_{k\to\infty} \mathrm{Cu}^{\sim}(\psi_k) = \lambda$. Even if $\lambda([e_C]) = [1_{\tilde{B}}]$ in $\mathrm{Cu}^{\sim}(\tilde{B})$ and $\varphi_k : C \to M_N(\tilde{B})$ is a sequence of homomorphisms such that $\lim_{k\to\infty} \mathrm{Cu}^{\sim}(\varphi_k) = \lambda$, and each $\psi_k(e_C)$ is a projection so that $[\psi_k(e_C)] = \lambda([e_C]) = [1_{\tilde{B}}]$ in $\mathrm{Cu}^{\sim}(\tilde{B})$, one may not have $\psi_k(e_C) \sim 1_{\tilde{B}}$ in $\mathrm{Cu}(\tilde{B})$. It is then impossible to perturb φ_k into homomorphisms $\psi_k : C \to \tilde{B}$ such that $\lim_{k\to\infty} \mathrm{Cu}^{\sim}(\psi_k) = \lambda$.

6 Unitization

The following is a result of L. Robert.

Lemma 6.1 (Lemma 3.2.1 of [30])**.** *Let A be a C^*-algebra of stable rank one and B be a unital C^*-algebra with finite stable rank. Let $e_A \in A$ be a strictly positive element. Let $\alpha : \mathrm{Cu}^\sim(A) \to \mathrm{Cu}^\sim(B)$ be a morphism in* **Cu** *such that $\alpha([e_A]) \le [1_B]$. Then there exists a unique morphism $\alpha^\sim : \mathrm{Cu}^\sim(\tilde{A}) \to \mathrm{Cu}^\sim(B)$ in* **Cu** *such that $\alpha([1_{\tilde{A}}]) = [1_B]$.*

Proof. We keep the same notation in the proof of Lemma 3.2.1 of [30]. For any $[a] \in W(\tilde{A})$ such that $[\pi(a)] = n < \infty$, one defines, for any integer m, (in $\mathrm{Cu}^\sim(B)$)

$$\alpha^\sim([a] - m[1_{\tilde{C}}]) = \alpha([a] - n[1_{\tilde{C}}]) + (n-m)[1_{\tilde{B}}]. \tag{e 6.1}$$

(Note that, by subsection 4.2 of [32], the revised definition of $\mathrm{Cu}^\sim(B)$ is the same as that defined in [30]). The exactly the same proof first shows that such $\alpha^\sim$ is uniquely defined, additive and sends positive elements to positive elements. Let $a_1, a_2 \in (\tilde{A} \otimes \mathcal{K})_+$ with $[a_1], [a_2] \in W(\tilde{A})$ be such that $[a_1] \le [a_2]$. We also use $\pi : \tilde{A} \to \mathbb{C}$ as in the proof of Lemma 3.2.1 of [30]. If $[\pi(a_1)] = [\pi(a_2)]$, as in the proof of Lemma 3.2.1 of [30], $\alpha^\sim([a_1]) \le \alpha^\sim([a_2])$. Consider now the case $[\pi(a_1)] < [\pi(a_2)]$. Let $1 > \varepsilon > 0$. Choose $0 < \delta < \varepsilon/8$ such that $\pi(f_{2\delta}(a_1)) = \pi(a_1)$ and $\pi(f_{2\delta}(a_1))$ is a projection. We may also assume that $\pi(f_{2\delta}(a_1)) < \pi(a_2)$, by replacing a_2 with $u^* g(a_2) u$ for some strictly positive functions in $C_0((0, \|a_2\|])$ and a scalar unitary $u \in \mathcal{K}^\sim$. Without loss of generality, we may further assume that $f_\delta(a_1) \in \mathrm{Her}(a_2)$ (see Proposition 2.4 of [33]). Choose $a_3 \le a_2$ such that $\pi(a_3) \perp \pi(f_{2\delta}(a_1))$ and $[\pi(a_3)] + [\pi(f_{2\delta}(a_1)] = [\pi(a_2)]$. Put $c = (1_{(A\otimes\mathcal{K})^\sim} - f_{\delta/2}(a_1))a_3(1_{(A\otimes\mathcal{K})^\sim} - f_{\delta/2}(a_1))$. Then $\pi(c) \perp \pi(f_{2\delta}(a_1))$ and $\pi(c) = \pi(a_3)$. Now $[c] \in W(\tilde{A})$ and

$$[(a_1 - \delta)_+] + [c] \le [a_2] \ \text{ and } \ [\pi(a_1-\delta)_+] + [\pi(c)] = [\pi(a_2)]. \tag{e 6.2}$$

Let $n_{1,\delta} = [\pi((a_1-\delta)_+)]$. Then, since $\alpha^\sim$ maps positive elements to positive elements and is additive, by (e 6.1) and (e 6.2), as in the proof of Lemma 3.2.1 of [30], one computes that

$$\alpha^\sim([(a_1-\delta)_+]) \le \alpha^\sim([(a_1-\delta)_+]) + \alpha^\sim([c]) = \alpha^\sim([(a_1-\delta)_+] + [c]) \tag{e 6.3}$$
$$\le \alpha([(a_1-\delta)_+] + [c] - n_{1,\delta}[1] - [\pi(c)][1]) + (n_{1,\delta} + [\pi(c)])[1] \le \alpha^\sim([a_2]). \tag{e 6.4}$$

As in the proof of Lemma 3.2.1 of [30], it follows that $\alpha^\sim$ preserves the order. One then proceeds the rest of the proof of Lemma 3.2.1 of [30]. ☐

Remark 6.2. In 6.8, it will be shown that there are homomorphisms $\varphi, \psi : A \to B$ such that $\mathrm{Cu}(\varphi) = \mathrm{Cu}(\psi)$ but $\mathrm{Cu}(\varphi^\sim) \ne \mathrm{Cu}(\psi^\sim)$. It may be worth noticing that Lemma 6.1 deals with a different situation.

Definition 6.3. Let F_1 and F_2 be two finite dimensional C^*-algebras. Suppose that there are (not necessary unital) homomorphisms $\varphi_0, \varphi_1 : F_1 \to F_2$. Define

$$A = A(F_1, F_2, \varphi_0, \varphi_1) = \{(f,g) \in C([0,1], F_2) \oplus F_1 : f(0) = \varphi_0(g) \ \text{ and } \ f(1) = \varphi_1(g)\}. \tag{e 6.5}$$

Denote by $\mathcal{C}$ the class of all C^*-algebras of the form $A = A(F_1, F_2, \varphi_0, \varphi_1)$. These C^*-algebras are called Elliott-Thomsen building blocks as well as one dimensional non-commutative CW complexes (see [13] and [14]).

Denote by $\mathcal{I}_0$ the subclass of C^*-algebras C in $\mathcal{C}$ such that $K_1(C) = \{0\}$.

All C^*-algebras in $\mathcal{C}$ have stable rank one (see, for example, Lemma 3.3 of [22]) and are semiprojective (see Theorem 6.22 of [13]).

Lemma 6.4. *Let $A, C \in \mathcal{I}_0$ be C^*-algebras such that there is an isomorphism $\varphi : A\otimes\mathcal{K} \cong C\otimes\mathcal{K}$. Then there exists an integer $n \ge 1$ and an injective homomorphism $\iota : \varphi(A) \to M_n(C)$ such that $\iota \circ \varphi(A)$ is a full C^*-subalgebra of $M_n(C)$ and $\mathrm{Cu}^\sim(\iota) = \mathrm{Cu}^\sim(\mathrm{id}_{\varphi(A)})$.*

(Note that we identify A with the first corner $A \otimes e_{1,1}$ of $A \otimes \mathcal{K}$.)

Proof. Let D be a liminal C^*-algebra. Denote by $\mathrm{Irr}(D)$ the set of irreducible representations of D. If $d \in D_+$ and $\xi \in \mathrm{Irr}(D)$, let us denote $r_\xi(d)$ the rank of $\xi(d)$ (with value in $\{0\}\cup\mathbb{N}\cup\{\infty\}$).

Let $e_A \in A^{\mathbf{1}}_+$ be a strictly positive element of A. There is $N \ge 1$ such that

$$1 \le \inf\{r_\xi(e_A) : \xi \in \mathrm{Irr}(A\otimes\mathcal{K})\} \le \sup\{r_\xi(e_A) : \xi \in \mathrm{Irr}(A\otimes\mathcal{K})\} \le N,$$

viewing A as a hereditary C^*-subalgebra of $A\otimes\mathcal{K}$. Put $A_1 = \varphi(A)$. Then A_1 is a full hereditary C^*-subalgebra of $C \otimes \mathcal{K}$ as isomorphisms preserve the full hereditary C^*-subalgebras. Hence $A_1 = \mathrm{Her}(\varphi(e_A))$. Note that, since φ is an isomorphism,

$$1 \le \inf\{r_\xi(\varphi(e_A)) : \xi \in \mathrm{Irr}(C\otimes\mathcal{K})\} \le \sup\{r_\xi(\varphi(e_A)) : \xi \in \mathrm{Irr}(C\otimes\mathcal{K})\} \le N.$$

Fix a strictly positive element $e_C \in C$. Then, there is $N_1 \ge 1$ such that

$$1 \le \inf\{r_\xi(e_C) : \xi \in \mathrm{Irr}(C)\} \le \sup\{r_\xi(e_C) : \xi \in \mathrm{Irr}(C)\} \le N_1.$$

Let $\{e_{i,j}\} \subset \mathcal{K}$ be a system of matrix units and $E_j = \sum_{i=1}^j e_{i,i}$ (for $j \ge 1$). Put $c_n := e_C \otimes E_n$. Then $r_\xi(c_n) = n \cdot r_\xi(e_C)$. Therefore there is an integer $n \ge 1$ such that $d_\tau(\varphi(e_A)) < d_\tau(c_n)$ for all $\tau \in T(C)$. Working in $\tilde{C}$ if C is not unital, by 3.18 of [22], $\varphi(e_A) \lesssim c_n$ in $\mathrm{Cu}(C)$. Note that $\varphi(A)$ is a full hereditary C^*-subalgebra $C \otimes \mathcal{K}$. Since $C \otimes \mathcal{K}$ has stable rank one, by Theorem 1.0.1 of [30], there is a homomorphism $\iota : \varphi(A) \to M_n(C)$ such that $\mathrm{Cu}^\sim(\iota) = \mathrm{id}_{\mathrm{Cu}^\sim(C)}$.

Note, if $\iota(c) = 0$ for some $c \in C_+$, then $\mathrm{Cu}^\sim(\iota)([c]) = 0$. Thus, ι is injective. To see $\iota\circ\varphi(A)$ is full, one needs to show that $\iota\circ\varphi(e_A)$ is full in $M_n(C)$. But $\iota\circ\varphi(e_A) \sim \varphi(e_A)$ and $\varphi(e_A)$ is full since φ is an isomorphism. □

We will use the following known and easy fact.

Lemma 6.5. *Let B be as in 4.2. Suppose that $b \in M_n(\tilde{B})_+$ is such that $[b]$ is a compact element in $\mathrm{Cu}^\sim(B)$. Then there is $g \in C_0((0, \|b\|])$ such that $g(b)$ is a projection.*

Proof. By Theorem 6.1 of [32] (recall B has stable rank at most 2), there is a projection $p \in M_N(\tilde{B})$ for some integer $N \ge 1$ such that $[b] = [p]$ in $\mathrm{Cu}^\sim(\tilde{B})$.

If 0 is not an isolated point of $\mathrm{sp}(b)$, for any $\varepsilon > 0$, there is a nonzero element $c \le b$ such that $c \perp f_\varepsilon(b)$. Since B is simple, $\tau(c) \ne 0$ for any $\tau \in T(B)$. It follows that

$$d_\tau(f_\varepsilon(b)) < d_\tau(p) \text{ for all } \tau \in T(B). \tag{e 6.6}$$

However, since p is compact, for all small ε, $[p] \le [f_\varepsilon(b)]$ in $\mathrm{Cu}(\tilde{B})^{\triangleq}$. This contradicts with (e 6.6). So 0 must be an isolated point of $\mathrm{sp}(b)$. Thus there is a such g so that $g(b)$ is a projection. □

Theorem 6.6. *Let C be a C^*-algebra in $\mathcal{I}_0$ with a strictly positive element e_C and B be a simple C^*-algebra which satisfies conditions in 4.2. Suppose that $\lambda : \mathrm{Cu}^\sim(\tilde{C}) \to \mathrm{Cu}^\sim(\tilde{B})$ is a morphism in* **Cu** *such that $\lambda([1_{\tilde{C}}]) = [b]$ for some (compact element) $b \in M_n(\tilde{B})_+$ (for some integer $n \ge 1$). Suppose also that there exists a sequence of homomorphisms $\varphi_k : C \to M_n(\tilde{B})$ such that $\lim_{k\to\infty} \mathrm{Cu}^\sim(\varphi_k) = \lambda|_{\mathrm{Cu}^\sim(C)}$.*

(1) If $\lambda([e_C])$ is not a compact element, then there exists a sequence of homomorphisms $\psi_k : \tilde{C} \to \overline{bM_n(\tilde{B})b}$ such that

$$\lim_{k\to\infty} \mathrm{Cu}^\sim(\psi_k) = \lambda.$$

(2) If $\lambda([e_C])$ is a compact element and $\lambda([c]) \neq 0$ for any $c \in C_+ \setminus \{0\}$, then there exists a sequence of homomorphisms $\psi_k : \tilde{C} \to M_n(\tilde{B})$ such that

$$\lim_{k\to\infty} \mathrm{Cu}^\sim(\psi_k) = \lambda.$$

Proof. Consider case (2) first. If $\lambda([e_C])$ is a compact element, then, for all sufficiently small $0 < \varepsilon < 1$,

$$\lambda([e_C]) \leq \lambda([f_\varepsilon(e_C)]) \leq \lambda([e_C]). \tag{e 6.7}$$

Let $g \in C_0((0, \|e_C\|])_+$ with the support in $(0, \varepsilon/2]$. Then $\lambda([g(e_C)]) = 0$. The assumption on λ implies that $g(e_C) = 0$. It follows that C is unital. Since $[e_C] = [1_C] \ll [1_C]$, this implies that $[\varphi_k(1_C)] = \lambda([1_C])$ (for all large k).

Let $e_0 := 1_{\tilde{C}} - 1_C$. By Lemma 6.5, we may assume that $b = p$ for some projection $p \in M_n(\tilde{B})$. If $\lambda([e_0]) = 0$, then $\lambda([1_C]) = \lambda([1_{\tilde{C}}])$. Define $\psi_k : \tilde{C} \to M_n(\tilde{B})$ by $\psi_k|_C = \varphi_k$ and $\psi_k(1_{\tilde{C}}) = \varphi_k(1_C)$. (Warning: we only have $[\psi_k(1_{\tilde{C}})] + 2[1_{\tilde{B}}] = [p] + 2[1_{\tilde{B}}]$ in $\mathrm{Cu}(\tilde{B})$ for large k.)

If $\lambda([e_0]) \neq 0$, then, for each k,

$$d_\tau(\varphi_k(1_C)) \leq d_\tau(p) \ \text{ for all } \ \tau \in T(\tilde{B}) \ \text{ and } \ d_\tau(\varphi_k(1_C)) < d_\tau(p) \ \text{ for all } \ \tau \in T(B). \tag{e 6.8}$$

It follows from Corollary A.4 of [17] that (since $\hat{p}$ is continuous on $T(B)$)

$$\varphi_k(1_C) \lesssim p \qquad \text{in } \mathrm{Cu}(\tilde{B}). \tag{e 6.9}$$

There is a partial isometry $v_k \in M_n(\tilde{B})$ such that $v_k v_k^* = \varphi_k(1_C)$ and $v_k^* v_k \leq p$. Define $\psi_k : \tilde{C} \to pM_n(A)p$ by $\psi_k(c) = v_k^* \varphi_k(c) v_k$ for all $c \in C$ and $\psi_k(1_{\tilde{C}}) = p$. Since C is unital, $\mathrm{Cu}^\sim(\psi_k|_C) = \mathrm{Cu}^\sim(\varphi_k)$. It follows from Lemma 6.1 that (2) holds.

For (1), we assume that $\lambda([e_C])$ is not a compact element. Again, by Lemma 6.5 and the fact $1_{\tilde{C}}$ is a projection, we may assume $b = p$ is a projection. By Lemma 5.3 ((ii) of (1)), we may assume that there is a sequence of homomorphisms φ_k which maps C into $pM_n(\tilde{B})p$ such that $\lim_{k\to\infty} \mathrm{Cu}^\sim(\varphi_k) = \lambda|_{\mathrm{Cu}^\sim(C)}$. Define $\psi_k : \tilde{C} \to pM_n(\tilde{B})p$ such that $\psi_k|_C = \varphi_k$ and $\psi_k(1_{\tilde{C}}) = p$. Then

$$\mathrm{Cu}^\sim(\psi_k|_C) = \mathrm{Cu}^\sim(\varphi_k) \ \text{ and } \ (\text{see Lemma 6.1}) \lim_{k\to\infty} \mathrm{Cu}^\sim(\psi_k) = \lambda.$$

□

The condition that $\lambda([c]) \neq 0$ for all $c \in C_+ \setminus \{0\}$ may be called "strictly positive".

Corollary 6.7. *Let C be a C^*-algebra in $\mathcal{I}_0$ with a strictly positive element e_C and B be a simple C^*-algebra which satisfies conditions in 4.2. Suppose that $\lambda : \mathrm{Cu}^\sim(C) \to \mathrm{Cu}^\sim(\tilde{B})$ is a morphism in* **Cu** *such that $\lambda([e_C]) \leq [e]$ for some nonzero projection $e \in M_n(\tilde{B})$ (for some $n \in \mathbb{N}$). Suppose also that there exists a sequence of homomorphisms $\varphi_k : C \to M_n(\tilde{B})$ such that $\lim_{k\to\infty} \mathrm{Cu}^\sim(\varphi_k) = \lambda$. Then there exists a sequence of homomorphisms $\psi_k : \tilde{C} \to M_n(\tilde{B})$ such that*

$$\lim_{k\to\infty} \mathrm{Cu}^\sim(\psi_k|_C) = \lambda.$$

Moreover, if $\lambda([e_C])$ is not a compact element, then we may require that $\psi_k(C) \subset eM_n(\tilde{B})e$.

Proof. Define $\psi_k|_C = \varphi_k$ and $\psi_k(1_{\tilde{C}}) = 1_n$. Then the first part of the statement follows. For the second part, we note that, since B is simple and stably proectionless, and $e \in M_n(\tilde{B})$ is a nonzero projection, e is a full element in $M_n(\tilde{B})$. It follows that $eM_n(\tilde{B})e \otimes \mathcal{K} \cong \tilde{B} \otimes \mathcal{K}$. Put $D = eM_n(\tilde{B})e$. By theorem 5.5 of [32], $\mathrm{Cu}^\sim(D) \cong \mathrm{Cu}^\sim(\tilde{B})$. Then the second part of the corollary follows from part (1) of Theorem 6.6.

□

Example 6.8. By Theorem 5.27 of [21], there is a separable simple stably projectionless C^*-algebra A with nontrivial $K_0(A)$ and with continuous scale such that $\ker\rho_A = K_0(A)$ and $A = \lim_{n\to\infty}(C_n, \varphi_n)$, where $C_n \in \mathcal{I}_0$ and $\varphi_{n,\infty} : C_n \to C$ is injective. Choose C_n so that ${\varphi_{n,\infty}}_{*0}(K_0(C_n)) \neq 0$. This also implies that $K_0(C_n) \neq \{0\}$. Note that, since A is stably projectionless, C_n is also stably projectionless.

Let $B := A \otimes \mathcal{W}$, where $\mathcal{W}$ is the unique separable amenable KK-contractible C^*-algebra with a unique tracial state (see [17]). Then B has continuous scale and $T(B) = T(A)$, and B is KK-contractible. By the classification theorem in [17], B is in fact a simple inductive limit of Razak algebras. Then, by Proposition 6.2.3 of [30],

$$\mathrm{Cu}^\sim(B) = \{0\} \sqcup \mathrm{LAff}_+^\sim(T(B)) = \{0\} \sqcup \mathrm{LAff}_+^\sim(T(A)) \text{ and } \mathrm{Cu}^\sim(A) = K_0(A) \sqcup \mathrm{LAff}_+^\sim(T(A)).$$

By Theorem 1.0.1 of [30], there is a homomorphism $j : A \to B$ such that

$$\mathrm{Cu}(j)|_{K_0(A)} = 0 \text{ and } \mathrm{Cu}^\sim(j)|_{\mathrm{LAff}_+^\sim(T(A))} = \mathrm{id}_{\mathrm{LAff}_+^\sim(T(A))}.$$

There is also a homomorphism $\iota : B \to A$ such that $\mathrm{Cu}^\sim(\iota)|_{\mathrm{LAff}_+^\sim(T(B))} = \mathrm{id}|_{\mathrm{LAff}_+^\sim(T(B))}$. Let $\psi := \iota \circ j \circ \varphi_{n,\infty} : C_n \to A$. Note $\mathrm{Cu}^\sim(\iota \circ j)|_{\mathrm{LAff}_+^\sim(T(A))} = \mathrm{id}_{\mathrm{LAff}_+^\sim(T(A))}$. Since C_n is stably projectionless, one has

$$\mathrm{Cu}(\psi) = \mathrm{Cu}(\varphi_{n,\infty}).$$

But, since $\psi_{*0} = 0$ and ${\varphi_{n,\infty}}_{*0} \neq 0$,

$$\mathrm{Cu}^\sim(\psi) \neq \mathrm{Cu}^\sim(\varphi_{n,\infty}) \text{ and } \mathrm{Cu}(\psi^\sim) \neq \mathrm{Cu}(\varphi_{n,\infty}^\sim).$$

7 Existence

Lemma 7.1. *Let A be a separable simple stably projectionless C^*-algebra with continuous scale such that $M_m(A)$ has almost stable rank one for all $m \geq 1$. Suppose also $QT(A) = T(A)$ and* $\mathrm{Cu}(A) = \mathrm{LAff}_+(T(A))$.

Then, for any **Cu** *morphism $\lambda : \mathrm{Cu}^\sim(C_0((0,1]))) \to \mathrm{Cu}^\sim(\tilde{A})$ with $\lambda([e_C]) \leq [a]$ for some $a \in M_n(\tilde{A})_+$ (for some integer $n \geq 1$), where e_C is a strictly positive element of $C_0((0,1])$, and $\lambda([c]) \neq 0$ for any $c \in C_0 \otimes \mathcal{K}_+ \setminus \{0\}$, there is a homomorphism $h : C_0((0,1]) \to M_n(\tilde{A})$ such that* $\mathrm{Cu}^\sim(h) = \lambda$.

Moreover, if $\lambda : \mathrm{Cu}^\sim(C_0((0,1]))) \to \mathrm{Cu}^\sim(A)$ with $\lambda([e_C]) \leq [a]$ for some $a \in M_n(A)_+$, then there exists a homomorphism $h : C_0((0,1])) \to M_n(A)$ such that $\mathrm{Cu}^\sim(h) = \lambda$.

Proof. Recall that A shares the same condition that B has in 4.2. Put $C_0 := C_0((0,1])$. Recall that $K_i(C_0) = \{0\}$, $i = 0, 1$. Note that, since C_0 has stable rank one, $\mathrm{Cu}(C_0)$ is orderly embedded into $\mathrm{Cu}^\sim(C_0)$. So $\mathrm{Cu}^\sim(C_0)_+ = \mathrm{Cu}(C_0)$ (see Lemma 3.1.2 of [30]). Note also that $\tilde{A}$ is unital and has stable rank at most 2 (see the proof of Theorem 6.13 of [32]). Thus λ maps $\mathrm{Cu}(C_0)$ to $\mathrm{Cu}(\tilde{A})^{\doteq}$ (see Lemma 4.4 and Corollary 4.13). Therefore it suffices to show that there is a homomorphism $h : C_0 \to M_n(\tilde{A})$ such that $\mathrm{Cu}(h) = \lambda|_{\mathrm{Cu}(C_0)}$.

Recall, by Theorem 4.4, that $\mathrm{Cu}(\tilde{A})^{\doteq} = (K_0(\tilde{A})_+ \setminus \{0\}) \sqcup \mathrm{LAff}_+(T(\tilde{A}))^\diamond$. Suppose that $\lambda([c])$ is compact for some non-zero $c \in (C_0 \otimes \mathcal{K})_+$. Note $[c] = \sup\{[f_{1/2^n}(c)] : n \in \mathbb{N}\}$. It follows, for some $n \geq 1$,

$$\lambda([c]) \leq \lambda([f_{1/2^n}(c)]). \tag{e 7.1}$$

However, since C_0 is stably projectionless, there is $c_0 \in \mathrm{Her}(c)_+ \setminus \{0\}$ such that $c_0 \perp f_{1/2^n}(c)$. By the assumption, $\lambda([c_0]) \neq 0$. This contradicts with (e 7.1) as C_0 has stable rank one. Hence $\lambda([c])$ is not compact for any $[c] \in C_0 \otimes \mathcal{K}_+ \setminus \{0\}$.

Thus $\lambda(\mathrm{Cu}(C_0)) \subset \mathrm{LAff}_+(T(\tilde{A}))^\diamond$ (see Theorem 6.1 of [32]).

It follows from Theorem 2.8 of [17] that there is a separable simple C^*-algebra A_1 which is an inductive limit of Razak algebras with continuous scale such that $T(A_1) = T(A)$. Note that $K_i(A_1) = \{0\}$, $i = 0, 1$. By Theorem A.26 of [17], there is an embedding $\iota : A_1 \to A$ which maps strictly positive elements to strictly positive elements such that ι induces an affine homeomorphism $\iota_T : T(A) \to T(A_1)$. Let $\iota^\sim : (A_1)^\sim \to \tilde{A}$ be the unital extension. We also write $\iota^\sim$ for the extension from $M_n(A_1^\sim)$ to $M_n(\tilde{A})$ for each integer $n \geq 1$. Thus $\iota^\sim$ induces an isomorphism $\iota^{\sharp\sim}$ form $\mathrm{LAff}_+(T(A_1^\sim))$ onto $\mathrm{LAff}_+(T(\tilde{A}))$. Since $A_1{}^\sim$ has stable rank one, it follows from Theorem 1.0.1 of [30] that there is a homomorphism $\varphi : C \to M_n(\tilde{A}_1)$ such that

$$\mathrm{Cu}^\sim(\varphi) = (\iota^{\sharp\sim})^{-1} \circ \lambda. \tag{e 7.2}$$

Define $h : C \to A$ by $h = \iota^\sim \circ \varphi$. Then $\mathrm{Cu}^\sim(h) = \lambda$.

For the "Moreover" part, we first note that $\mathrm{Cu}(A) = \mathrm{LAff}_+(T(A))$, as A is stably projectionless. Therefore, working in $\mathrm{Cu}(A)$, the above argument also works and produces a homomorphism $\varphi : C \to M_n(A)$ such that $\mathrm{Cu}(\varphi) = \lambda$. This part also follows from [31].

□

Definition 7.2. Let $\mathcal{A}_0$ be the family of C^*-algebras in $\mathcal{I}_0$ which consists of one C^*-algebra $C_0((0,1])$. A C^*-algebra A is in $\mathcal{A}_n$ if $A \in \mathcal{I}_0$ and, if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, or, if $A = \tilde{B}$, or if $\tilde{A} = B$ for some $B \in \mathcal{A}_{n-1}$, $n = 1, 2, ...$

Theorem 7.3. *Let $C \in \mathcal{I}_0$ be a C^*-algebra and let A be a separable simple stably projectionless C^*-algebra with continuous scale such that $M_m(A)$ has almost stable rank one for all $m \geq 1$. Suppose also $QT(A) = T(A)$ and $\mathrm{Cu}(A) = \mathrm{LAff}_+(T(A))$. Let e_C and e_A be strictly positive elements of C and A, respectively.*

(1) Suppose that there is a morphism $\lambda : \mathrm{Cu}^\sim(C) \to \mathrm{Cu}^\sim(A)$ in **Cu** *such that $\lambda([e_C]) \leq n[e_A]$ for some integer $n \geq 1$ and $\lambda([c]) \neq 0$ for any $c \in C_+ \setminus \{0\}$. Then there is an integer $m \geq n$ and a sequence of homomorphisms $\varphi_k : C \to M_m(A)$ such that $\lim_{k\to\infty} \mathrm{Cu}^\sim(\varphi_k) = \lambda$.*

(2) Also, if there is a morphism $\lambda : \mathrm{Cu}^\sim(C) \to \mathrm{Cu}^\sim(\tilde{A})$ in **Cu** *such that $\lambda([e_C]) \leq n[1_{\tilde{A}}]$ and $\lambda([c]) \neq 0$ for all $c \in (C \otimes \mathcal{K})_+ \setminus \{0\}$, then there exists an integer $m \geq n$ and a sequence of homomorphisms $\varphi_k : C \to M_m(\tilde{A})$ such that $\lim_{k\to\infty} \mathrm{Cu}^\sim(\varphi_k) = \lambda$.*

Proof. It follows from (the proof of) Proposition 5.2.2 of [30] that, $C \in \mathcal{A}_m$ for some $m \geq 0$. By Lemma 7.1, the lemma holds for any C^*-algebra $C \in \mathcal{A}_0$ and any A which meets the requirement of the lemma.

Assume that lemma holds for any C^*-algebra C in $\mathcal{A}_{m-1}$. It suffices to show that the lemma holds for any C^*-algebra C in $\mathcal{A}_m$ and any A as described in the lemma. Fix $C \in \mathcal{A}_m$.

Case (I) : Suppose that $h : C \otimes \mathcal{K} \to B \otimes \mathcal{K}$ is an isomorphism for some $B \in \mathcal{A}_{m-1}$. In situation (2), suppose that $\lambda : \mathrm{Cu}^\sim(C) \to \mathrm{Cu}^\sim(\tilde{A})$ is a morphism in **Cu** such that $\lambda([e_C]) \leq n[1_{\tilde{A}}]$.

By Lemma 6.4, there is an injective homomorphism $\iota : h(C) \to M_L(B)$ for some integer $L \geq 1$ such that $\mathrm{Cu}^\sim(\iota) = \mathrm{Cu}^\sim(\mathrm{id}_{h(C)})$. Since $B \in \mathcal{A}_{m-1}$, by the inductive assumption, there exists an integer $m_0 \geq n$ and a sequence of homomorphisms $\psi_k : M_L(B) \to M_{Lm_0}(\tilde{A})$ such that

$$\lim_{k\to\infty} \mathrm{Cu}^\sim(\psi_k) = \lambda \circ \mathrm{Cu}^\sim(h^{-1}).$$

Define $\varphi_k : C \to M_{Lm_0}(\tilde{A})$ by $\varphi_k(c) = \psi_k \circ \iota \circ h(c)$ for all $c \in C$. It follows that

$$\lim_{k\to\infty} \mathrm{Cu}^\sim(\varphi_k) = \lambda.$$

In situation (1), λ maps $\mathrm{Cu}^\sim(C)$ to $\mathrm{Cu}^\sim(A)$, then the argument above also works (but ψ_k maps $M_L(B)$ into $M_{Lm_0}(A)$).

Case (II): $C = \tilde{B}$ for some $B \in \mathcal{A}_{m-1}$. Note that C is unital and A is stably projectionless. Hence $\lambda : \mathrm{Cu}^{\sim}(C) \to \mathrm{Cu}^{\sim}(\tilde{A})$. (We do not need to consider the case $\lambda : \mathrm{Cu}^{\sim} \to \mathrm{Cu}^{\sim}(A)$.) Let $e_B \in B$ be a strictly positive element. Note that $\mathrm{Cu}^{\sim}(B)$ is orderly embedded into $\mathrm{Cu}^{\sim}(\tilde{B})$ (see Proposition 3.1.6 of [30]). Since $B \in \mathcal{A}_{m-1}$ and $\lambda([e_B]) \le \lambda([e_C]) \le n[1_{\tilde{A}}]$, by the inductive assumption, there is an integer $m_0 \ge n$ and a sequence of homomorphisms $\psi_k : B \to M_{m_0}(\tilde{A})$ such that

$$\lim_{k\to\infty} \mathrm{Cu}^{\sim}(\psi_k) = \lambda|_{\mathrm{Cu}^{\sim}(B)}. \tag{e 7.3}$$

If $\lambda([e_B])$ is not compact, we apply part (1) of Theorem 6.6 to obtain the desired maps φ_k. If $\lambda([e_B])$ is compact, since λ is strictly positive, by (2) of Theorem 6.6, there is also a sequence of homomorphisms $\varphi_k : C = \tilde{B} \to M_{nm_0}(\tilde{A})$ such that

$$\lim_{k\to\infty} \mathrm{Cu}^{\sim}(\varphi_k) = \lambda. \tag{e 7.4}$$

Case (3): $\tilde{C} = B$ for some $B \in \mathcal{A}_{m-1}$. Let $\lambda : \mathrm{Cu}^{\sim}(C) \to \mathrm{Cu}^{\sim}(\tilde{A})$ be such that $\lambda([e_C]) \le n[1_{\tilde{A}}]$. By Lemma 6.1, there is an extension $\lambda^{\sim} : \mathrm{Cu}^{\sim}(\tilde{C}) \to \mathrm{Cu}^{\sim}(\tilde{A})$ in **Cu** such that $\lambda^{\sim}|_{\mathrm{Cu}^{\sim}(C)} = \lambda$ and $\lambda(1_{\tilde{C}}) = (n+1)[1_{\tilde{A}}]$. Consider the following splitting short exact sequence (see Proposition 3.1.6 of [30]):

$$0 \to \mathrm{Cu}^{\sim}(C) \to \mathrm{Cu}^{\sim}(\tilde{C}) \overset{\mathrm{Cu}^{\sim}(\pi_{\mathbb{C}}^{C})}{\to} \mathrm{Cu}^{\sim}(\mathbb{C}) \to 0, \tag{e 7.5}$$

where $\pi_{\mathbb{C}}^{C} : \tilde{C} \to \mathbb{C}$ is the quotient map (and its extension). Let $a \in (\tilde{C} \otimes \mathcal{K})_+ \setminus \{0\}$. If $\pi_{\mathbb{C}}^{C}(a) = 0$, then $\lambda^{\sim}([a]) = \lambda([a]) \ne 0$. If $\pi_{\mathbb{C}}^{C}(a) \ne 0$, then, by the definition, $\lambda^{\sim}([a]) \ne 0$. Thus $\lambda^{\sim}([a]) \ne 0$ for any $a \in (\tilde{C} \otimes \mathcal{K})_+ \setminus \{0\}$. Since $B \in \mathcal{A}_{m-1}$, by the assumption, there exists a sequence of homomorphisms $h_k : B = \tilde{C} \to M_L(\tilde{A})$ for some $L \ge n$ such that $\lim_{k\to\infty} \mathrm{Cu}^{\sim}(h_k) = \lambda^{\sim}$. Choose $\varphi_k := h_k|_C$. Then $\lim_{k\to\infty} \mathrm{Cu}^{\sim}(\varphi_k) = \lambda$.

If $\lambda : \mathrm{Cu}^{\sim}(C) \to \mathrm{Cu}^{\sim}(A)$ with $\lambda([e_C]) \le n[e_A]$, then, since $\mathrm{Cu}^{\sim}(A) \to \mathrm{Cu}^{\sim}(\tilde{A})$ is an order embedding, by Theorem 5.3 of [32], one may view $\lambda : \mathrm{Cu}^{\sim}(C) \to \mathrm{Cu}^{\sim}(\tilde{A})$. It follows from Lemma 6.1, there is an extension $\lambda^{\sim} : \mathrm{Cu}^{\sim}(\tilde{C}) \to \mathrm{Cu}^{\sim}(\tilde{A})$ such that $\lambda^{\sim}|_{\mathrm{Cu}^{\sim}(C)} = \lambda$ and $\lambda^{\sim}([1_{\tilde{C}}]) = (n+1)[1_{\tilde{A}}]$. As proved above, $\lambda^{\sim}$ is strictly positive, i.e., $\lambda^{\sim}([c]) \ne 0$ for any $c \in (\tilde{C} \otimes \mathcal{K})_+ \setminus \{0\}$. Since $B \in \mathcal{A}_{m-1}$, there exists a sequence of homomorphisms $h_k : B = \tilde{C} \to M_L(\tilde{A})$ such that $\lim_{k\to\infty} \mathrm{Cu}^{\sim}(h_k) = \lambda^{\sim}$. Define $\varphi_k = h_k|_C$. Then $\lim_{k\to\infty} \mathrm{Cu}^{\sim}(\varphi_k) = \lambda$.

This completes the induction. Theorem follows. □

Corollary 7.4. *Let $C \in \mathcal{I}_0$ be a C^*-algebra with a strictly positive element e_C and let A be a finite separable simple stably projectionless C^*-algebra which is $\mathcal{Z}$-stable with continuous scale such that $QT(A) = T(A)$. Suppose that there is a morphism $\lambda : \mathrm{Cu}^{\sim}(C) \to \mathrm{Cu}^{\sim}(A)$ in* **Cu** *such that $\lambda([e_C]) \le [a]$ for some $a \in A_+$ and $\lambda([c]) \ne 0$ for all $c \in C_+ \setminus \{0\}$. Then there exists a sequence of homomorphisms $\varphi_k : C \to \overline{aAa}$ such that $\lim_{k\to\infty} \mathrm{Cu}^{\sim}(\varphi_k) = \lambda$.*

Moreover, there exists a sequence of injective homomorphisms $\varphi_k : C \to \overline{aAa}$ such that $\lim^{w}_{k\to\infty} \mathrm{Cu}^{\sim}(\varphi_k) = \lambda$.

Proof. Recall that A satisfies the condition that B satisfies in 4.2. Let $e_A \in A$ be a strictly positive element. Then $\widehat{[e_A]}$ is continuous on $T(A)$. It follows from Theorem 7.3 that there exists an integer $m \ge 2$ and a sequence of homomorphisms $\psi_k : C \to M_m(A)$ such that

$$\lim_{k\to\infty} \mathrm{Cu}^{\sim}(\psi_k) = \lambda. \tag{e 7.6}$$

Since A is stably projectionless and λ is strictly positive, $\lambda([e_C])$ is not compact. Applying (2) of Lemma 5.3, we obtain a sequence of homomorphisms $\varphi'_k : C \to A$ such that

$$\lim_{k\to\infty} \mathrm{Cu}^{\sim}(\varphi'_k) = \lambda. \tag{e 7.7}$$

To see the last part of the statement and to make homomorphisms injective, for each $k \geq 1$, choose $0 < \varepsilon_k < 1/2^{k+1}$ and define $L_k : C \to A_k := \mathrm{Her}(f_{\varepsilon_k}(a))$ by

$$L_k(c) = f_{\varepsilon_k}(a)\varphi_k(c)f_{\varepsilon_k}(a) \text{ for all } c \in C. \tag{e 7.8}$$

Since C is semiprojective, by choosing small ε_k, one obtains a homomorphism $\varphi_k'' : C \to A_k$ such that (see also Theorem 5.2)

$$\lim_{k\to\infty} \|\varphi_k''(c) - \varphi_k(c)\| = 0 \text{ for all } c \in C \text{ and} \tag{e 7.9}$$

$$\lim_{k\to\infty} \mathrm{Cu}^{\sim}(\varphi_k'') = \lambda. \tag{e 7.10}$$

Choose a nonzero function in $g_k \in C_0((0,1])_+$ with support in $(0, \varepsilon_k/3)$ and nowhere zero in $(0, \varepsilon_k/3)$. Put $B_k = \mathrm{Her}(g_k(a))$. Since A is stably projectionless, we may assume that B_k is nonzero. Note also $B_k \perp A_k$. Put

$$\sigma_k := \sup\{d_\tau(g_k) : \tau \in T(A)\} > 0. \tag{e 7.11}$$

Since A has continuous scale, we have that

$$\lim_{k\to\infty} \sigma_k = 0. \tag{e 7.12}$$

Note that B_k is a hereditary C^*-subalgebra of A and therefore it is also $\mathcal{Z}$-stable (Cor. 3.1 of [39]). Choose a nonzero hereditary C^*-subalgebra $D_k \subset B_k$ which has continuous scale (see Remark 5.3 of [16]). By Theorem 6.11 of [32], $\mathrm{Cu}^{\sim}(D_k) = K_0(D_k) \sqcup \mathrm{LAff}_+^{\sim}(T(D_k))$. By Corollary A.8 of [17] and Theorem 4.1 of [21], there exists $\tau_0 \in T(D_k)$ such that $\rho_{D_k}(x)(\tau_0) = 0$ for all $x \in K_0(D_k)$, where $\rho_{D_k} : K_0(D_k) \to \mathrm{Aff}(T(D_k))$ is the usual paring. Recall $\mathcal{W}$ is the unique separable KK-contractible amenable simple $\mathcal{Z}$-stable C^*-algebra with a unique tracial state τ_W. Define $\gamma : \mathrm{Cu}^{\sim}(D_k) \to \mathrm{Cu}^{\sim}(\mathcal{W})$ by $\gamma|_{K_0(D_k)} = 0$ and $\gamma(f)(\tau_W) = rf(\tau_0)$ for all $f \in \mathrm{LAff}_+^{\sim}(\tilde{T}(D_k))$ for a choice of $0 < r < 1$. Recall $\mathrm{Cu}^{\sim}(D_k) = \mathrm{Cu}^{\sim}(A)$. Note that $\gamma \circ \lambda([c]) \neq 0$ for all $c \in C_+ \setminus \{0\}$. We choose r so that $\gamma \circ \lambda([e_C])(\tau_W) < 1$. By Theorem 1.0.1 of [30], there is an injective homomorphism (since $\gamma \circ \lambda$ is strictly positive) $h_k : C \to \mathcal{W}$ such that $\mathrm{Cu}^{\sim}(h_k) = \gamma \circ \lambda$. Let E be a separable KK-contractible amenable simple $\mathcal{Z}$-stable C^*-algebra with $T(E) = T(D_k)$ and has stable rank one (see Theorem 2.8 of [17]). Let $h_{E,D} : E \to D_k$ be a nonzero homomorphism given by Theorem A.26 of [17] so that $h_{E,D}$ induces the identification of $T(E) = T(D)$. Let $\eta : \mathrm{Cu}^{\sim}(\mathcal{W}) \to \mathrm{Cu}^{\sim}(E)$ be defined by $\eta(f)(\tau) = f(\tau_W)$ for all $f \in \mathrm{LAff}_+^{\sim}(\tilde{T}(\mathcal{W}))$. Applying Theorem 1.0.1 of [30] again, there is a monomorphism $h_{W,E} : \mathcal{W} \to E$ such that $\mathrm{Cu}^{\sim}(h_{W,E}) = \eta$.

Define $h_{k,C,D} := h_{E,D} \circ h_{W,E} \circ h_k : C \to D_k$. Then $h_{k,C,D}$ is an injective homomorphism. Define $\varphi_k : C \to \mathrm{Her}(a)$ by $\varphi_k(c) = \varphi_k'(c) + h_{k,C,D}(c)$ for all $c \in C$. Recall that $D_k \perp B_k$. The map φ_k is injective. It remains to show that $\lim_{k\to\infty}^{w} \mathrm{Cu}^{\sim}(\varphi_k) = \lambda$.

Since $h_{k,C,D}$ factors through $\mathcal{W}$, $\mathrm{Cu}^{\sim}(h_{k,C,D})|_{K_0(C)} = 0$. Note here we view $K_0(A)$ as a subset of $\mathrm{Cu}^{\sim}(C)$ (see subsection 6.1 and Theorem 6.1 of [32]). Then, by (e 7.10), for any finite subset $G \subset \mathrm{Cu}^{\sim}(C)$, there exists $N \geq 1$ such that, for any $k \geq N$ (see also 5.1),

$$\mathrm{Cu}^{\sim}(\varphi_k)(x) = \lambda(x) \text{ for all } x \in G \cap K_0(C). \tag{e 7.13}$$

Let $f, g \in G$, $f \ll g$ be such that neither f nor g are compact. Recall, by Theorem 5.3 of [32], that $\mathrm{Cu}^{\sim}(A)$ is orderly embedded into $\mathrm{Cu}^{\sim}(\tilde{A})$. Let $\lambda^{\sim} : \mathrm{Cu}^{\sim}(\tilde{C}) \to \mathrm{Cu}^{\sim}(\tilde{A})$ be the unique extension of λ given by Lemma 6.1. As in the proof of case (3) in the proof of Theorem 7.3, $\lambda^{\sim}$ is strictly positive.

Let $\bar{f}$ and $\bar{g}$ be as in the proof of 5.2 with $\|\bar{f}\|, \|\bar{g}\| \le 1$. We also retain other notations in the proof 5.2 related to f and g.

Since f and g are not compact elements, by (ii) of Theorem 6.1 of [32], neither are $\bar{f}$ and $\bar{g}$. Since $\lambda^{\sim}$ is strictly positive, $\lambda^{\sim}(\bar{f})$ and $\lambda^{\sim}(\bar{g})$ are not compact. Let $d_f, d_g \in M_r(\tilde{A})_+$ (for some $r \ge 1$) such that $[d_f] = \lambda^{\sim}(\bar{f})$, and $[d_g] = \lambda^{\sim}(\bar{g})$. Note, as $\lambda^{\sim}$ is the unique extension of λ,

$$\lambda^{\sim}(\bar{f}) = \lambda(f) + m_f[1_{\tilde{A}}] + m_g[1_{\tilde{A}}] \text{ and } \lambda^{\sim}(\bar{g}) = \lambda(g) + m_g[1_{\tilde{A}}] + m_f[1_{\tilde{A}}] \tag{e 7.14}$$

(see (e 6.1)). Then (recall that $[d_g]$ cannot be represented by a projection), there is $0 < \delta < 1/2$ such that

$$\pi_{\mathbb{C}}^C(f_\delta(d_g)) = \pi_{\mathbb{C}}^C(d_g), \;\; d_\tau(f_{\delta/2}(d_g)) > \tau(f_\delta(d_g)) \text{ for all } \tau \in T(A) \text{ and} \tag{e 7.15}$$
$$[\bar{f}] \ll [f_{2\delta}(d_g)] \le [d_g]. \tag{e 7.16}$$

Thus, by (e 7.10), there exists an integer $N_1 \ge 1$ such that, for $k \ge N_1$,

$$[\varphi_k'^{\sim}(\bar{f})] \le [f_{2\delta}(d_g)]. \tag{e 7.17}$$

Therefore (see also (e 7.15))

$$d_\tau(\varphi_k'^{\sim}(\bar{f})) < \tau(f_\delta(d_g)) < \tau(f_{\delta/2}(d_g)) \le d_\tau(f_{\delta/2}(d_g)) \le d_\tau(\bar{g}) \text{ for all } \tau \in T(A) \text{ and} \tag{e 7.18}$$
$$d_\tau(\varphi_k'^{\sim}(\bar{f})) \le \tau(f_\delta(d_g)) \le \tau(f_{\delta/2}(d_g)) \le d_\tau(f_{\delta/2}(d_g)) \le d_\tau(\bar{g}) \text{ for all } \tau \in T(\tilde{A}). \tag{e 7.19}$$

Note that the lower semicontinuous function $[\widehat{f_{\delta/2}(d_g)}] - \widehat{f_\delta(d_g)}$ is strictly positive on the compact set $T(A)$. It follows that

$$\eta := \inf\{d_\tau(f_{\delta/2}(d_g)) - \tau(f_\delta(d_g)) : \tau \in T(A)\} > 0. \tag{e 7.20}$$

Note that we may assume that $\bar{f} \in M_{r+m_g}(\tilde{C})$ (see the lines below (e 5.11) and lines below (e 5.8) in the proof of 5.2). We may also assume, for all $k \ge N_1$,

$$(r + m_g)\sigma_k < \eta/4. \tag{e 7.21}$$

For any $1/2 > \varepsilon_0 > 0$, write

$$f_{\varepsilon_0}(\bar{f}) = S + c_{f,\varepsilon_0}, \tag{e 7.22}$$

where $S \in M_{r+m_g}(\mathbb{C})_+$ and $c_{f,\varepsilon_0} \in M_{r+m_g}(C)_{s.a.}$ and $\|S\| \le 1$ and $\|c_{f,\varepsilon_0}\| \le 2$. Recall (identifying S with the scalar matrix),

$$\varphi_k'^{\sim}(f_{\varepsilon_0}(\bar{f})) = S + \varphi_k'(c_{f,\varepsilon_0}) \text{ and } \varphi_k^{\sim}(f_{\varepsilon_0}(\bar{f})) = S + \varphi_k'^{\sim}(c_{f,\varepsilon_0}) + h_{k,C,D}(c_{f,\varepsilon_0}). \tag{e 7.23}$$

We estimate that, by (e 7.11), for all $\tau \in T(A)$.

$$|\tau(h_{k,C,D}((c_{f,\varepsilon_0})))| \le 2(r + m_g)\sigma_k < \eta/2. \tag{e 7.24}$$

Combining this with (e 7.23), (e 7.18), and (e 7.20), we obtain, for any $1/2 > \varepsilon_0 > 0$, if $k \ge N_1$,

$$d_\tau(\varphi_k^{\sim}(f_{\varepsilon_0}(\bar{f}))) < d_\tau(d_g) \text{ for all } \tau \in T(A) \text{ and} \tag{e 7.25}$$
$$d_\tau(\varphi_k^{\sim}(f_{\varepsilon_0}(\bar{f}))) \le d_\tau(d_g) \text{ for all } \tau \in T(\tilde{A}). \tag{e 7.26}$$

It follows from Theorem 4.12, if $k \ge N_1$ (in $\mathrm{Cu}(\tilde{A})$), $[\varphi_k^{\sim}(f_{\varepsilon_0}(\bar{f}))] \le [d_g]$. Since N_1 does not depend on ε_0, this implies that (in $\mathrm{Cu}(\tilde{A})$) $\mathrm{Cu}(\varphi_k^{\sim})(\bar{f}) \le [d_g]$. In other words (see also (e 7.14) and the lines below (e 5.8)), if $k \ge N_1$,

$$\mathrm{Cu}^{\sim}(\varphi_k)(f) + (m_f + m_g + 2)[1_{\tilde{A}}] = [\varphi_k(a^f)] + m_g[1_{\tilde{A}}] \le \lambda(g) + (m_g + m_f + 2)[1_{\tilde{A}}] \tag{e 7.27}$$

(recall that A has stable rank at most 2). Thus, if $k \geq N_1$,

$$\mathrm{Cu}^{\sim}(\varphi_k)(f) \leq \lambda(g). \tag{e 7.28}$$

The same argument shows that, if $k \geq N_1$,

$$\lambda(f) \leq \mathrm{Cu}^{\sim}(\varphi_k(g)). \tag{e 7.29}$$

Hence, combining with the last two displays and (e 7.13), one obtains

$$\lim^w_{n\to\infty} \mathrm{Cu}^{\sim}(\varphi_k) = \lambda.$$

□

Definition 7.5. Let C be a separable C^*-algebra such that $T(C) \neq \emptyset$ and $QT(C) = T(C)$. Let B be a separable simple C^*-algebra with continuous scale such that $QT(B) = T(B)$ and $\mathrm{Cu}(B) = \mathrm{LAff}_+(T(B))$. Let $\lambda : \mathrm{Cu}^{\sim}(C) \to \mathrm{Cu}^{\sim}(B)$ be a morphism in **Cu** such that $\lambda([e_C]) \leq [e_B]$, where $0 \leq e_C \leq 1$ and $0 \leq e_B \leq 1$ are strictly positive elements of C and B, respectively. Let $T_0(C)$ and $T_0(B)$ be the sets of all traces on C and B with norm no more than 1, respectively.

Let $a \in C_+$ with $\|a\| \leq 1$. For each n consider $x_n = \sum_{k=1}^{2^n} (1/2^n) p_{(t_{k,n},1]}$, where $t_{k,n} = k/2^n$ and $p_{(t_{k,n},1]}$ is the open spectral projection of a associated with the $(t_{k,n}, 1]$ in C^{**}. Note that $d_\tau((a - t_{k,n})_+) = \tau(p_{(t_{k,n},1]})$ for all $\tau \in T_0(C)$ and $\tau(x_n) = \sum_{k=1}^{2^n} (1/2^n) d_\tau((a - t_{k,n})_+)$ for all $\tau \in T_0(C)$. Moreover,

$$\sup\{|\tau(x_n) - \tau(a)| : \tau \in T_0(C)\} \leq 1/2^n. \tag{e 7.30}$$

For each $s \in T_0(B)$, define, for each $a \in M_r(C)_+$ (for integer $r \geq 1$),

$$\lambda_T(s)(a) = \int_0^\infty \lambda([(a - t)_+])\widehat{\ }(s)dt. \tag{e 7.31}$$

By Proposition 4.2 of [18], $\lambda_T(s)$ defines a lower semi-continuous quasitrace on $C \otimes \mathcal{K}$. Note that B has continuous scale. So $e_B \in \mathrm{Ped}(B)$. Since $\lambda([e_C]) \leq [e_B]$, if $a \in M_r(C)_+$, $\lambda([(a-t)_+])\widehat{\ }(s) \leq r\|a\|$ for all $t \in [0, \|a\|]$ and $s \in T(B)$. Since $QT(C) = T(C)$, $\lambda_T(s)$ is in $T_0(C)$. Proposition 4.2 of [18] also implies that the map $s \mapsto \lambda_T(s)$ is the affine continuous map from $T_0(B)$ to $T_0(C)$ induced by λ. Note

$$\lambda_T(s)(a) = \lim_{n\to\infty} (\sum_{k=1}^{2^n} (1/2^n)\lambda([(a - t_{k,n})_+])\widehat{\ }(s)). \tag{e 7.32}$$

Moreover, for $a \in C_+$ with $\|a\| \leq 1$,

$$\lim_{n\to\infty} \sup\{|\lambda_T(s)(a) - (\sum_{k=1}^{2^n} (1/2^n)\lambda([(a - t_{k,n})_+])\widehat{\ }(s))| : s \in T(B)\} = 0. \tag{e 7.33}$$

Let $\varphi : C \to A$ be a homomorphism. Then, for any $a \in C_+$ with $\|a\| \leq 1$,

$$\lim_{n\to\infty} \sup\{|\tau(\varphi(c)) - \sum_{k=1}^{2^n} (1/2^n)\tau(\varphi(f_{1/2^{n+3}}(a - t_{k,n})_+))| : \tau \in T(B)\} = 0. \tag{e 7.34}$$

Now suppose that B is stably projectionless and λ is strictly positive. If $\varphi_k : C \to B$ is a sequence of injective homomorphisms such that $\lim^w_{n\to\infty} \mathrm{Cu}^{\sim}(\varphi_k) = \lambda$, then, for each fixed $a \in C_+$ with $\|a\| \leq 1$ and $n > 1$, there is $N \geq 1$ such that, when $j \geq N$,

$$\lambda([(a - t_{k,n})_+]) \leq [\varphi_j(f_{1/2^{n+2}}((a - t_{k+1,n})_+)] \tag{e 7.35}$$

$$\leq [\varphi_j(f_{1/2^{n+3}}((a - t_{k+1,n})_+)] \leq \lambda([a - t_{k+1,n})_+), \;\; k = 1, 2, ..., 2^n. \tag{e 7.36}$$

It follows that, for all $\tau \in T(B)$,

$$\lambda([(a-t_{k,n})_+])\widehat{\ }(\tau) \le \tau(\varphi_j(f_{1/2^{n+3}}((a-t_{k+1,n})_+) \le \lambda([a-t_{k+1,n})_+)\widehat{\ }(\tau). \tag{e 7.37}$$

By (e 7.34), (e 7.33), and (e 7.37),

$$\lim_{k\to\infty} \sup\{|\tau(\varphi_k)(a) - \lambda_T(\tau)(a)| : \tau \in T_0(B)\} = 0. \tag{e 7.38}$$

Recall that if A is a finite exact separable simple $\mathcal{Z}$-stable C^*-algebra then A satisfies the conditions in the following statement.

Theorem 7.6. *Let $C = \lim_{n\to\infty} C_n$ with a strictly positive element e_C, where each $C_n \in \mathcal{I}_0$ and each map $\iota_n : C_n \to C_{n+1}$ is injective, and let A be a separable simple C^*-algebra with continuous scale and with a strictly positive element e_A such that $M_m(A)$ has almost stable rank one for all $m \ge 1$ and $QT(A) = T(A)$, and $\mathrm{Cu}(A) = \mathrm{LAff}_+(T(A))$. Suppose that $\lambda : \mathrm{Cu}^\sim(C) \to \mathrm{Cu}^\sim(A)$ is a morphism in* **Cu** *such that $\lambda([e_C]) \le [e_A]$ and $\lambda([c]) \ne 0$ for any $c \in C_+ \setminus \{0\}$. Then there exists a sequence of contractive completely positive linear maps $L_n : C \to A$ and a sequence of injective homomorphisms $h_n : C_n \to A$ such that*

$$\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0 \ \textit{for all}\ a, b \in C,$$
$$\textit{and, for each fixed } m,\ \lim_{n\to\infty} \|L_n(\iota_{m,\infty}(c)) - h_n(\iota_{m,n}(c))\| = 0 \ \text{for all}\ c \in C_m,$$
$$\text{and}\ \lim_{n\to\infty} \sup_{\tau\in T(A)} \|\tau(L_n(a)) - \lambda_T(\tau)(a)\| = 0 \ \textit{for all}\ a \in C.$$

Proof. We first assume that A is stably projectionless. For each k, consider $\alpha_k := \lambda \circ \mathrm{Cu}^\sim(\iota_{k,\infty})$. By Corollary 7.4, there exists a sequence of injective homomorphisms $\varphi_{k,n} : C_k \to A$ such that $\lim^w_{n\to\infty} \mathrm{Cu}^\sim(\varphi_{k,n}) = \alpha_k$. Then (see (e 7.38))

$$\lim_{n\to\infty} \sup\{|\tau \circ \varphi_{k,n}(c) - \alpha_{kT}(\tau)(c)| : \tau \in T_0(A)\} = 0 \ \text{for all}\ c \in C_k. \tag{e 7.39}$$

One obtains a sequence of injective homomorphisms $h_n : C_n \to A$ and, since C is amenable, a sequence of contractive completely positive linear maps $L_n : C \to A$ such that, for any fixed m,

$$\lim_{n\to\infty} \|L_n(\iota_{m,\infty}(c)) - h_n(\iota_{m,n}(c))\| = 0 \ \text{for all}\ c \in C_m,$$
$$\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0 \ \text{for all}\ a, b \in C \ \text{and}$$
$$\lim_{n\to\infty} \sup_{\tau\in T(B)} \|\tau(L_n(a)) - \lambda_T(\tau)(a))\| = 0 \ \text{for all}\ a \in C.$$

If A is not stably projectionless, by Proposition 2.2, A has stable rank one. Then, by Theorem 1.0.1 of [30], there is a homomorphism $H : C \to A$ such that $\mathrm{Cu}^\sim(H) = \lambda$. Choose $L_n = H$ and $h_n = H \circ \iota_{n,\infty}$. Then this case also follows. □

Corollary 7.7. *Let $C = \lim_{n\to\infty}(C_n, \iota_n)$ be as in Theorem 7.6 which is simple and has continuous scale and A be a finite exact separable simple stably projectionless $\mathcal{Z}$-stable C^*-algebra with continuous scale. Suppose that there is an isomorphism*

$$\Gamma : (K_0(C), T(C), r_C) \cong (K_0(A), T(A), r_A). \tag{e 7.40}$$

Then there exists a sequence of contractive completely positive linear maps $L_n : C \to A$ and a sequence of injective homomorphisms $h_n : C_n \to A$ such that

$$\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0 \ \text{for all}\ a, b \in C, \tag{e 7.41}$$

$$\lim_{n\to\infty} \sup_{\tau\in T(B)} \|\tau(L_n(a)) - \lambda_T(\tau)(a)\| = 0 \ \text{for all}\ a \in C \tag{e 7.42}$$

$$\textit{and, for each fixed } m,\ \lim_{n\to\infty} \|L_n(\iota_{m,\infty}(c)) - h_n(\iota_{m,n}(c))\| = 0 \ \text{for all}\ c \in C_m, \tag{e 7.43}$$

$$\lim_{n\to\infty} \sup_{\tau\in T(B)} \|\tau(h_n(\iota_{m,n}(c))) - \lambda_T(\tau)(\iota_{m,\infty}(c))\| = 0 \ \text{for all}\ c \in C_m, \tag{e 7.44}$$

where $\lambda_T : T(A) \to T(C)$ *is the affine homeomorphism given by* Γ.

Proof. By Proposition 6.2.3 of [30], $\mathrm{Cu}^{\sim}(C) = K_0(C) \sqcup \mathrm{LAff}_+^{\sim}(T(C))$. Also, by Theorem 6.1.1 of [32] (see also subsection 6.3 of [32]), $\mathrm{Cu}^{\sim}(A) = K_0(A) \sqcup \mathrm{LAff}_+^{\sim}(T(A))$. Let e_C and e_A be strictly positive elements of C and A, respectively. By (e 7.40), there is an isomorphism $\lambda : \mathrm{Cu}^{\sim}(C) \to \mathrm{Cu}^{\sim}(A)$ (in **Cu**) such that $\lambda([e_C]) = [e_A]$. Thus, the corollary follows from Theorem 7.6 immediately. □

Corollary 7.7 plays an important role in achieving the following theorem which was first proved with the additional condition that A has stable rank one. The only place where we need the condition that A has stable rank one was to have a homomorphism $h : C \to A$, where $C = \lim_{n\to\infty}(C_n, \iota_n)$, $C_n \in \mathcal{I}_0$ and $\iota_n : C_n \to C_{n+1}$ are injective, C has continuous scale, and $(K_0(C), T(C), r_C) = (K_0(A), T(A), r_A)$ such that $[h]$ induces the identification map on $(K_0(C), T(C), r_C)$. Note the identification map on the invariant set gives a strictly positive morphism $\lambda : \mathrm{Cu}^{\sim}(C) \to \mathrm{Cu}^{\sim}(A)$ with $\lambda([e_C]) = [e_A]$, where e_C and e_A are strictly positive elements of C and A, respectively. So the existence of such h follows from Theorem 1.0.1 of [30]. In fact, one only needs an approximate version of Robert's result. Without assuming A has stable rank one, one may not apply the result of L. Robert. However, one can apply Corollary 7.7 to obtain a sequence of homomorphisms h_k that approximates λ which improves the original version of the following theorem.

Theorem 7.8 (Theorem 7.12 of [21])**.** *Let A be separable amenable simple stably projectionless C^*-algebra with continuous scale such that $T(A) \neq \{0\}$ and satisfying the UCT. Then $A \otimes Q$ has generalized tracial rank one.*

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